What makes the harmonic series so special?

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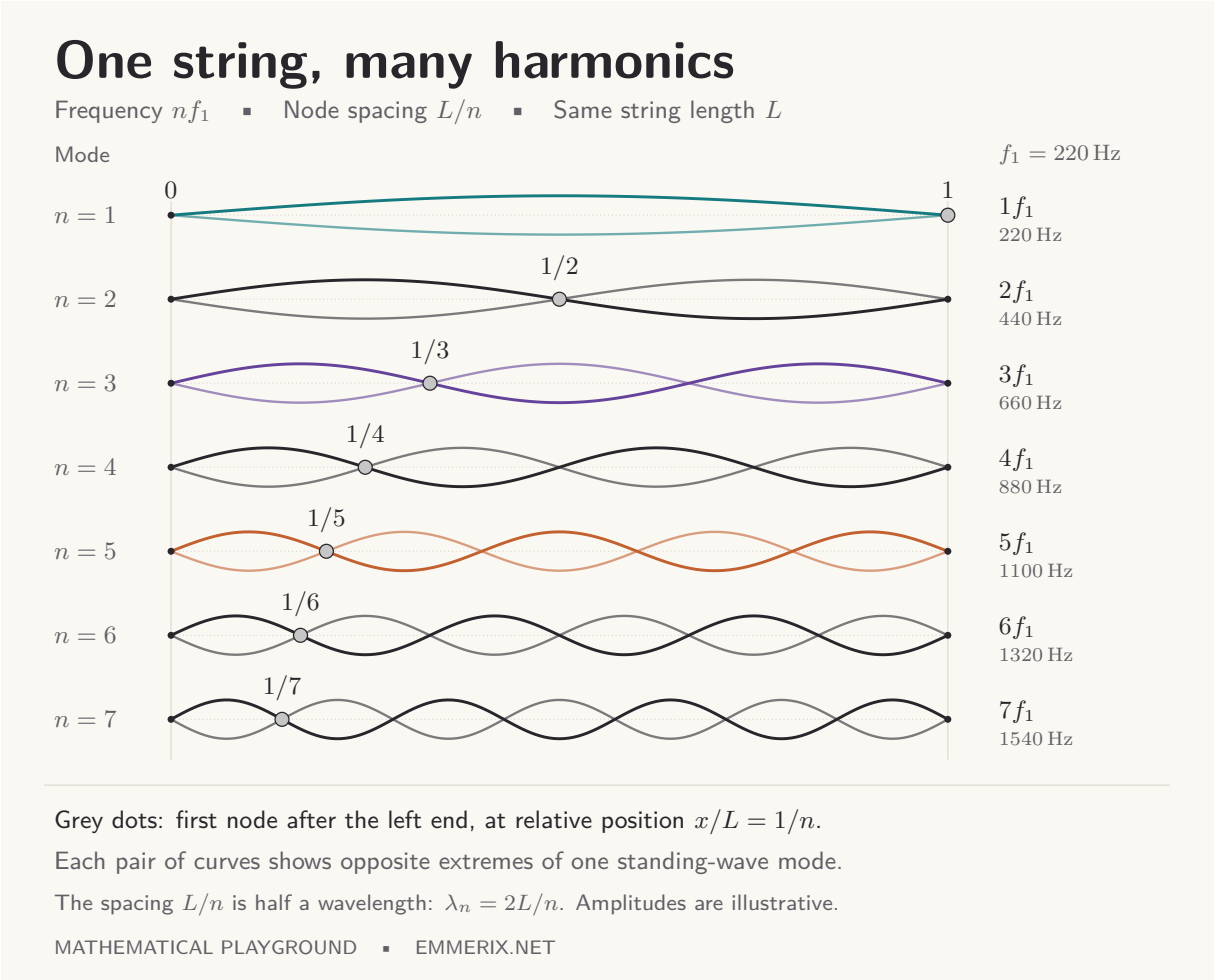

One string, many harmonic frequencies. Each row shows a standing-wave mode of the same string: more arches mean a higher frequency and smaller spacing between neighbouring nodes. The grey dots mark the reciprocal length ratios 1, 1/2, …, 1/7.

Finite sums, explicit bounds, and the arithmetic that remains visible.

I find the harmonic series fascinating because so many mathematical questions meet in this deceptively simple sum. Here I want to explore it from an arithmetic point of view: staying with exact rational numbers wherever possible and working with finite sums rather than passing to infinite limits. This keeps the factors and cancellations in each partial sum visible, helping us investigate the series' special properties and its connections with prime numbers, the atoms of arithmetic. Passing to a limit can "squeeze out" arithmetic information: distinct rational sequences can share a limit while having very different prime-factor patterns. How much can we learn by following the fractions themselves, one step at a time?

The word *harmonic* has a musical connection (Kullman, 2001). A string fixed at both ends can vibrate in the opening picture: one arch, two arches, three arches, and so on. Their frequencies are $f_1, 2f_1, 3f_1, \ldots$. Their node spacings go the other way: $L, L/2, L/3, \ldots$. Divide those spacings by L, and our familiar fractions appear. Several modes can sound together; their vibrations add to form a superposition. This musical connection makes the name intuitive, while adding the positive fractions themselves leads to our mathematical question.

Why is it called harmonic?

David E. Kullman's *What's Harmonic about the Harmonic Series?* (2001) gives a more precise answer to the naming question. The reciprocals form a *harmonic progression*: each term after the first is the harmonic mean of its two neighbours. For positive numbers a, b, define $\mathrm{HM}(a, b) = 2/(1/a + 1/b)$. Then, for $n \geq 2$,

$$\mathrm{HM}\Big(\frac{1}{n-1}, \frac{1}{n+1}\Big) = \frac{2}{(n-1)+(n+1)} = \frac{1}{n}.$$

Kullman connects this mean to ancient Greek musical ratios. He credits Oresme with the earliest recorded divergence proof, around 1350, but reports a much later occurrence of the name: the Oxford English Dictionary cites Chambers's *Cyclopedia* (1727–51). We should therefore distinguish the history of the proof from the history of the terminology.

What makes the harmonic series so special? First, we will show that it sits exactly at the boundary between convergence and divergence for sums of reciprocal powers. We revisit the classic proof of Nicole Oresme, later Bishop of Lisieux, and generalize it slightly. The same simple idea explains why reciprocal squares settle down, reciprocal square roots keep growing, and the harmonic series is just on the divergent side of the boundary.

Second, we will highlight and prove an arithmetic property of the reciprocal progression behind the harmonic series. Splitting it into the **"Evens" and the "Odds"** reveals a surprising rule about factors of two: the even-denominator terms reproduce a scaled harmonic sum, while the odd-denominator terms follow an exact rule for their reduced numerators. The prime clockwork makes those factors visible.

The harmonic numbers already visited this playground in *Of Autumn Leaves and Coupon Collectors*. Here we follow the fractions themselves, one step at a time.

Oresme's boxes and the boundary at 1

The harmonic partial sum is

$$H_N = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}.$$

Its terms become smaller. That alone does not tell us whether their total stays bounded. In the fourteenth century, Nicole Oresme found a beautifully simple way to settle the question: group the terms into blocks of lengths 1, 2, 4, 8, and so on.

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots .$$

The first 1 stands on its own. Each following block ends at a power of two. Replace every fraction in a block by its smallest member:

Denominators	Number of terms	Smallest term	Block contributes at least
2	1	1/2	1/2
3, 4	2	1/4	$2/4 = 1/2$
5 through 8	4	1/8	$4/8 = 1/2$
9 through 16	8	1/16	$8/16 = 1/2$

Every box contains at least another half. After m boxes, the last denominator is $N = 2^m$, and

$$H_{2^m} \geq 1 + \frac{m}{2}.$$

For Oresme's life beyond these boxes, see the Stanford biography. Julian Havil's *Gamma: Exploring Euler's Constant* (Princeton University Press, 2003) is a companion for further exploration.

Keep the boxes. Change the exponent.

To compare the harmonic series with reciprocal squares and reciprocal square roots, we keep Oresme's boxes and change the exponent. We begin with finite sums and explicit bounds. For real $s > 1$, their limit is the Riemann zeta function $\zeta(s)$.

Let s be a positive *real* number, and define the finite partial zeta sum

$$Z_N(s) = \sum_{n=1}^{N} \frac{1}{n^s}.$$

For $s = 1$ this is H_N. For $s = 2$ we add reciprocal squares. For $s = 1/2$ we add reciprocal square roots. The square-root case serves as an algebraic comparison; the harmonic numbers themselves remain rational. Every $Z_N(s)$ is a finite number; convergence asks whether these totals approach a finite limit as the cutoff grows.

Number the boxes starting at $r = 0$. Box r begins just after 2^r and ends at 2^{r+1}:

$$B_r(s) = \sum_{n=2^r+1}^{2^{r+1}} \frac{1}{n^s}.$$

It contains exactly 2^r terms. Each is at least $2^{-(r+1)s}$ and strictly less than 2^{-rs}. Multiply by the number of terms:

$$2^{-s}\, 2^{r(1-s)} \le B_r(s) < 2^{r(1-s)}.$$

This one inequality does the work. Doubling gives twice as many terms, while the individual term scale falls by 2^s. The scale of a whole box therefore changes by the factor

$$q = \frac{2}{2^s} = 2^{1-s}.$$

The actual block sums need not have ratio exactly q; their lower and upper bounds do. Those bounds are enough to decide convergence.

Exponent	Factor q	Block bounds
$s = 2$	$1/2$: shrinking	$2^{-r}/4 \le B_r(2) < 2^{-r}$
$s = 1$	1: constant scale	$1/2 \le B_r(1) < 1$
$s = 1/2$	$\sqrt{2}$: growing	$2^{r/2}/\sqrt{2} \le B_r(1/2) < 2^{r/2}$

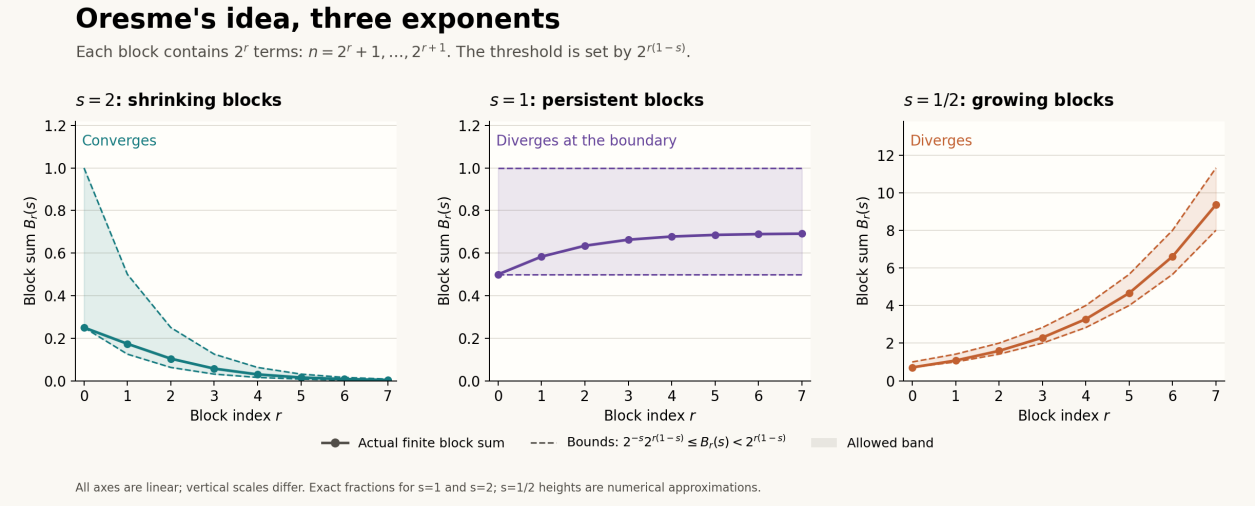

Figure 1: Each dot is one complete dyadic box (between consecutive powers of two), not an individual term and not a running total sum. The index $r = 0$ means denominator 2; $r = 1$ means denominators 3 and 4; $r = 2$ means 5 through 8. Shading marks the proved bounds. Vertical scales differ. Integer-exponent sums were computed as exact fractions; square-root heights are numerical approximations.

Above 1: even the whole remaining tail becomes small

Suppose $s > 1$. Then $0 < q < 1$. The upper bounds form a shrinking geometric progression. For any finite number $m \geq 1$ of boxes,

$$Z_{2^m}(s) < 1 + \frac{1 - q^m}{1 - q} < 1 + \frac{1}{1 - q}.$$

The partial sums increase, because every term is positive, but cannot cross this fixed ceiling. Any cutoff lies below some power of two, so the ceiling holds for every cutoff. An increasing, bounded sequence of real numbers has a finite limit. We have proved convergence for *every* real $s > 1$, not only for the exponent 2.

We can do better than a ceiling: we can bound everything still to come. Start at *any* positive integer N and group the tail using endpoints $N, 2N, 4N, 8N, \ldots$. Its box r contains $2^r N$ terms, each smaller than $(2^r N)^{-s}$. Its total is therefore smaller than $N^{1-s} q^r$.

If $M > N$ is any finite endpoint, choose enough of these boxes to cover it. A finite geometric sum gives

$$0 < Z_M(s) - Z_N(s) < \frac{N^{1-s}}{1 - 2^{1-s}}.$$

Only now, if we want a name for the limit, write it as $\zeta(s)$. The finite estimate supplies the explicit error bound

$$0 < \zeta(s) - Z_N(s) \leq \frac{N^{1-s}}{1 - 2^{1-s}} \qquad (s > 1).$$

At $s = 2$, the bound is $2/N$. At $s = 3/2$, it is $N^{-1/2}/(1 - 2^{-1/2})$. Both tend to zero. The proof needs only counting and a finite geometric sum.

At 1: exactly enough to prevent convergence

At $s = 1$, the box factor is $q = 1$. Oresme's lower bound now comes with an upper bound:

$$1 + \frac{m}{2} \leq H_{2^m} < 1 + m.$$

Every doubling adds between one half and one. However small the increase in the exponent, moving above 1 restores convergence; at 1 itself the series still diverges.

Below 1: now the boxes grow

For $0 < s < 1$, we have $q > 1$. The lower bound $2^{-s}q^r$ grows from box to box. Divergence is immediate: even a single sufficiently late box exceeds any prescribed amount.

For a quantitative estimate, take $N = 2^m$, with $m \geq 1$. The finite geometric total is

$$G_m = 1 + q + \cdots + q^{m-1} = \frac{N^{1-s} - 1}{2^{1-s} - 1}.$$

Our bounds say simply

$$1 + 2^{-s}G_m \leq Z_N(s) < 1 + G_m.$$

For the reciprocal square roots, $s = 1/2$, this becomes

$$1 + \frac{\sqrt{N} - 1}{2 - \sqrt{2}} \leq Z_N(1/2),$$

$$Z_N(1/2) < 1 + \frac{\sqrt{N} - 1}{\sqrt{2} - 1}.$$

So these totals grow on the scale of $\sqrt{N}$, while the harmonic sums grow on the scale of the *number of doublings*. These bounds do not by themselves determine the precise leading coefficient of the square-root growth.

The boundary, in one sentence

For real exponents, the positive defining zeta series converges precisely when $s > 1$; it diverges when $s \leq 1$. We have proved every positive case with the same boxes. For $s \leq 0$, each term is at least 1, so divergence is even simpler.

The boundary is at the *exponent* $s = 1$: there is no finite series value $\zeta(1)$. This is a statement about the positive defining series. Analytic continuation gives another meaning to $\zeta(s)$ away from its convergence region; in particular, its finite value at $s = 1/2$ is not the limit of the positive partial sums studied here. The Riemann critical line concerns that continuation and is a separate question.

Sharper finite bounds by telescoping

A second finite comparison reveals more about the size of the sums. The derivation is optional; it also gives an alternative convergence proof for reciprocal squares.

Optional: derive the sharper bounds

The boxes locate the convergence boundary. For reciprocal square roots and reciprocal squares, telescoping gives sharper finite estimates: the leading factor 2 for the former and a smaller tail bound for the latter. A *telescoping sum* is one in which consecutive differences cancel, leaving only the endpoints.

Reciprocal square roots. Rationalizing a difference gives $\sqrt{n+1}-\sqrt{n} = 1/(\sqrt{n+1}+\sqrt{n})$. Comparing the denominators with $2\sqrt{n}$, for $n \geq 1$, gives

$$2(\sqrt{n+1}-\sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n}-\sqrt{n-1}).$$

Sum the lower bound from 1 through N. For the upper bound, keep the first term as 1 and sum from 2 through N. All the intermediate square roots disappear:

$$2(\sqrt{N+1}-1) < Z_N(1/2) \leq 2\sqrt{N}-1.$$

The upper equality occurs only at $N = 1$. These bounds prove divergence and identify the leading growth as $2\sqrt{N}$. Indeed, at every finite cutoff, $1 \leq 2\sqrt{N} - Z_N(1/2) < 2$. Dividing the bounds by $\sqrt{N}$ gives, only at the final classification step, $Z_N(1/2)/\sqrt{N} \to 2$.

The harmonic boundary. Keep Oresme's boxes. If $2^k \leq N < 2^{k+1}$, the completed boxes below N and the next full endpoint give

$$1 + \frac{k}{2} \leq H_N < k + 2.$$

Here growth counts doublings. The lower bound increases without any finite ceiling, even though it does so much more slowly than the square-root sum.

Reciprocal squares. For $n \geq 2$, place each term between two fractions that telescope:

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n}.$$

For *any two finite integers* $M > N \geq 1$, adding these inequalities gives

$$\frac{1}{N+1} - \frac{1}{M+1} < Z_M(2) - Z_N(2) < \frac{1}{N} - \frac{1}{M}.$$

No matter how far we extend the sum, the extra amount is less than $1/N$. Thus the partial sums satisfy the Cauchy condition: sufficiently late sums are uniformly close to one another. Completeness of the real numbers then gives a finite limit. This sharpens our earlier $2/N$ bound and supplies an alternative proof of convergence at $s = 2$, using neither a geometric series nor an integral.

The prime clockwork remembers the factors

The prime clockwork is a way to grow an arithmetic picture one integer at a time. **One column belongs to each prime, one row to each positive integer, and three tables show three different readings.** The construction is developed in my 2026 arXiv paper, *The Prime Clockwork: A Dynamic Representation of Modular and Multiplicative Arithmetic*. The earlier posts *Prime Clockwork* and *On Prime Vectors and Unique Factorization of Rationals* introduce the picture.

Think of row n as time n. Each installed prime p has its own clock. The seconds counter $R_p(n)$ cycles through $0, 1, \ldots, p-1$. Every return to zero rings the clock and increments its minutes counter $M_p(n)$. Thus minutes count completed multiples of p; seconds record what is left over. The valuation $V_p(n)$ is a third reading: the exponent of p in n. Minutes and valuation are different: at $n = 12, p = 2$, they read 6 and 2.

$$n = pM_p(n) + R_p(n), \qquad 0 \le R_p(n) < p.$$

Consequently $M_p(n) = \lfloor n/p \rfloor$ and $R_p(n) = n \bmod p$. These formulas describe the readings. To generate them, the clock only needs a common $+1$ **tick**, comparison, reset, and carry:

$$(R_p(n+1), M_p(n+1)) = \begin{cases} (R_p(n)+1, M_p(n)), & R_p(n) < p-1, \\ (0, M_p(n)+1), & R_p(n) = p-1. \end{cases}$$

Each column performs this update autonomously, without consulting another prime column. The array itself grows: start at row 1 with no clocks. After each tick, if none of the installed clocks rings, the new integer becomes a new clock period, starting with seconds 0 and minutes 1. These new periods are exactly the primes: a composite integer would have triggered a clock belonging to a smaller prime divisor.

Finally, read the valuation from the same column's history. A nonzero seconds reading means that p does not divide n. At a ring, $n = pM_p(n)$, so strip off one factor of p and consult the earlier row addressed by the minutes counter:

$$V_p(n) = \begin{cases} 0, & R_p(n) \neq 0, \\ 1 + V_p(M_p(n)), & R_p(n) = 0. \end{cases}$$

This rule applies once the clock is installed, with $V_p(j) = 0$ for $1 \le j < p$. Since $M_p(n) < n$, the lookup always points backwards. The extra 1 is added to the value at the *minutes address*, not to the valuation in the preceding row. A valuation alone is therefore not the complete state of the advancing clock.

Table 1 — Valuation exponents $V_p(n)$. Read across a row to recover the integer's prime factorization. Read down one column to follow that prime's multiplicity.

	Prime p				
n	2	3	5	7	11
1	·	·	·	·	·
2	1	·	·	·	·
3	0	1	·	·	·
4	2	0	·	·	·
5	0	0	1	·	·
6	1	1	0	·	·
7	0	0	0	1	·
8	3	0	0	0	·
9	0	2	0	0	·
10	1	0	1	0	·
11	0	0	0	0	1
12	2	1	0	0	0

Table 2 — Minutes counters $M_p(n)$. Each entry counts the completed multiples $p, 2p, \ldots$ up to row n. It increases by one whenever that column rings.

	Prime p				
n	2	3	5	7	11
1	·	·	·	·	·
2	1	·	·	·	·
3	1	1	·	·	·
4	2	1	·	·	·
5	2	1	1	·	·
6	3	2	1	·	·
7	3	2	1	1	·
8	4	2	1	1	·
9	4	3	1	1	·
10	5	3	2	1	·
11	5	3	2	1	1
12	6	4	2	1	1

Table 3 — Seconds counters $R_p(n)$. These are the modular remainders. A zero is a divisibility alarm; the next tick restarts that column's cycle at 1.

	Prime p				
n	2	3	5	7	11
1	·	·	·	·	·
2	0	·	·	·	·
3	1	0	·	·	·
4	0	1	·	·	·
5	1	2	0	·	·
6	0	0	1	·	·
7	1	1	2	0	·
8	0	2	3	1	·
9	1	0	4	2	·
10	0	1	0	3	·
11	1	2	1	4	0
12	0	0	2	5	1

Tables 1, 2 and 3 use the same row and column order. A dot means *not yet installed*, rather than a numerical zero. For example, the 11-column appears only at row 11. In a completed static factorization table, its earlier valuation entries can be filled with zero.

Watch the step from 8 to 9. In the 2-column, seconds/minutes change from $(0, 4)$ to $(1, 4)$, and the valuation drops from 3 to 0. In the 3-column they change from $(2, 2)$ to $(0, 3)$: the clock rings, and its new valuation is $1 + V_3(3) = 2$. The two columns obey the same tick, each using its own period and history.

From integer rows to harmonic-number rows

To extend the valuation coordinates to harmonic sums, for a positive rational $a = A/B$, subtract the denominator's exponent vector from the numerator's. Give the columns the order $2, 3, 5, 7, 11, \ldots$:

$$v_p(A/B) = V_p(A) - V_p(B), \qquad P(a) = (v_2(a), v_3(a), v_5(a), \ldots).$$

These signed coordinates already record the reduced rational value. Positive entries put primes in the numerator; negative entries put them in the denominator. Multiplication adds coordinates. Addition requires a new numerator calculation; it does not add prime vectors coordinate by coordinate.

Table 4 — The harmonic numbers in the same prime coordinates. Now let row n carry the whole partial sum $H_n = 1 + 1/2 + \cdots + 1/n = A_n/B_n$, rather than just the integer n. Each prime still has its own column, but its entry is now the signed exponent $v_p(H_n) = V_p(A_n) - V_p(B_n)$: positive for a numerator factor, negative for a denominator factor, and zero if the prime is absent from the reduced fraction.

Addition can introduce numerator primes larger than n, so this table includes every prime needed for these twelve sums. All unlisted prime coordinates are zero. The bold 2-column shows the dyadic staircase.

n	H_n	2	3	5	7	11	13	61	97	137	509	761	863	7129
1	1	**0**	0	0	0	0	0	0	0	0	0	0	0	0
2	$\frac{3}{2}$	**-1**	+1	0	0	0	0	0	0	0	0	0	0	0
3	$\frac{11}{6}$	**-1**	-1	0	0	+1	0	0	0	0	0	0	0	0
4	$\frac{25}{12}$	**-2**	-1	+2	0	0	0	0	0	0	0	0	0	0
5	$\frac{137}{60}$	**-2**	-1	-1	0	0	0	0	0	+1	0	0	0	0
6	$\frac{49}{20}$	**-2**	0	-1	+2	0	0	0	0	0	0	0	0	0
7	$\frac{363}{140}$	**-2**	+1	-1	-1	+2	0	0	0	0	0	0	0	0
8	$\frac{761}{280}$	**-3**	0	-1	-1	0	0	0	0	0	0	+1	0	0
9	$\frac{7129}{2520}$	**-3**	-2	-1	-1	0	0	0	0	0	0	0	0	+1
10	$\frac{7381}{2520}$	**-3**	-2	-1	-1	+2	0	+1	0	0	0	0	0	0
11	$\frac{83711}{27720}$	**-3**	-2	-1	-1	-1	0	0	+1	0	0	0	+1	0
12	$\frac{86021}{27720}$	**-3**	-2	-1	-1	-1	+2	0	0	0	+1	0	0	0

Read row 6: $H_6 = 49/20 = 2^{-2}5^{-1}7^2$. The 3-coordinate has changed from -1 in $H_5 = 137/60$ to zero: adding $1/6 = 10/60$ gives $147/60 = 49/20$. The factor 3 cancels, while 7^2 appears in the numerator. Already $H_3 = 11/6$ needs the prime 11, even though only three summands have been added.

The harmonic rows advance by the exact rule $H_{n+1} = H_n + 1/(n+1)$. For a chosen prime, the refinement rule below keeps enough modular information to determine its new exponent. The 2-column is particularly simple: at each +1 tick, its displayed exponent stays fixed unless the new index is a power of two, when it drops by one.

Technical detail: refinement without prefix replay

The sums themselves obey $H_{n+1} = H_n + 1/(n+1)$. To generate their prime columns, advance to $j = n + 1$ and let each prime work independently. Normalize by $X_p(j) = p^{\ell_p(j)} H_j$, where $\ell_p(j) = \lfloor \log_p j \rfloor$. The internal state retains a remainder $x = X_p(j) \bmod p^R$, not just the displayed exponent. Here rational remainders use modular inverses of denominators coprime to p.

This rule computes a column for a *specified* prime p. Discovering additional numerator primes is a separate task; they need not be among the primes installed by the integer clocks at that row.

For $j \geq p$, write $j = pm + r$, $0 \leq r < p$, and $k = \ell_p(j)$. Separate denominators divisible by p from the others:

$$U_p(m,r) = \sum_{\substack{1\le a\le pm+r\\ p\nmid a}} \frac{1}{a}, \qquad X_p(j) = X_p(m) + p^k U_p(m,r).$$

The minutes address $m = \lfloor j/p \rfloor$ points to an earlier state in the same column. If $x \neq 0$, display the exact exponent $v_p(x) - k$. If $x = 0$, increase R and refine only the cached ancestry $j, \lfloor j/p \rfloor, \lfloor j/p^2 \rfloor, \ldots$; the base rows $j < p$ use a short direct sum. Only when $R > k$ is the correction needed, modulo p^{R-k}. The finite power-sum evaluator in my report, *Oresme Blocks and Prime-Valuation Columns of the Harmonic Numbers*, subsection "The minutes hand: refinement without prefix replay", computes it without rescanning the previous harmonic rows. One tick means one new output row; extra precision can require extra work.

The 2-column is exceptionally simple. Here $X_2(j) \equiv 1 \pmod 2$ for every row, so $R = 1$ always suffices and no refinement is needed. Start the displayed exponent at zero; keep it unchanged at each $+1$ tick unless the new index is a power of two, when it drops by one. The 3-column already needs more information to resolve cancellations, as rows 6 and 7 show.

The two-adic staircase and the Evens and Odds

Why the harmonic two-coordinate is a staircase

The bold 2-column in Table 4 reads $0, -1, -1, -2, -2, -2, -2, -3, \ldots$. The exact rule is $v_2(H_n) = -\lfloor \log_2 n \rfloor$. A short common-denominator argument explains why no factor of two is ever cancelled.

Choose the integer k so that $2^k \le N < 2^{k+1}$. Use the least common multiple of the denominators as a common denominator:

$$D_N = \operatorname{lcm}(1, 2, \ldots, N).$$

It contains exactly k factors of two. Among the integers D_N/j, for $j = 1, \ldots, N$, exactly one is odd: $D_N/2^k$. Indeed, 2^k is the only denominator in this range containing k factors of two; dividing by any other denominator leaves at least one factor of two.

One odd integer plus any number of even integers is odd. Consequently, in

$$H_N = \frac{\sum_{j=1}^{N} D_N/j}{D_N},$$

the numerator is odd. Reduction can cancel odd factors, but none of the denominator's k factors of two. Thus $v_2(H_N) = -k$, as claimed.

What the Evens leave to the Odds

Here is a small reward for keeping that arithmetic visible. Separate the first $2N$ harmonic terms into odd and even denominators:

$$U_N = \sum_{j=1}^{N} \frac{1}{2j-1}, \qquad H_{2N} = U_N + \frac{1}{2} H_N.$$

The first prime coordinate follows the exact rule

$$v_2(U_N) = 2v_2(N).$$

Count the trailing binary zeros of N, then double the count: that is the number of twos in the reduced numerator of U_N. For $N = 1, 2, 3, 4$, the sums are $1, 4/3, 23/15, 176/105$, with two-exponents $0, 2, 0, 4$. Since all denominators are odd, no factor of two can hide downstairs.

Compare this with the harmonic staircase: at $N = 2^k$, the harmonic two-coordinate is $-k$, while the odd-sum coordinate is $2k$. Both sums increase in ordinary size; the coordinates describe their factors of two.

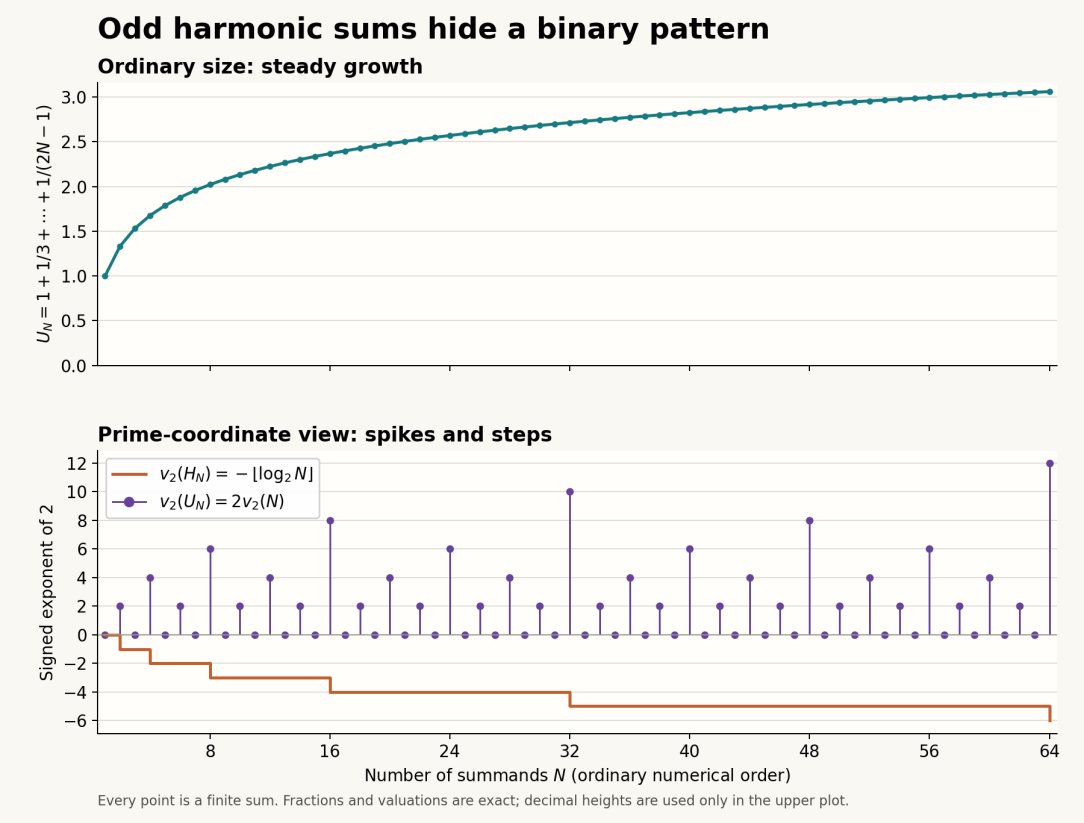

Figure 2: A second view of finite sums, for $N = 1$ through 64. The upper panel shows ordinary magnitude; the lower panel shows just the prime coordinate belonging to 2. Divisibility structure survives even when a broad growth description looks smooth.

The odd-sum identity is a known result, discussed on MathOverflow in 2016, with proofs by Will Sawin and Dongryul Kim. A finite proof is included below for readers who want to follow this extra trail.

Optional: a finite proof of the odd-sum valuation

The table suggests the rule; the following argument proves it for every positive integer N. We use congruences of fractions with odd denominators. Modulo a power of two, every odd denominator has an inverse: for example, $1/3 \equiv 3 \pmod 8$, because $3 \cdot 3 \equiv 1 \pmod 8$. We can therefore calculate with these fractions using ordinary modular arithmetic.

Put $k = v_2(N)$, $M = 2^{k+1}$, and $x_j = 2j - 1$. Reversing the denominators sends x_j to $2N - x_j$. Averaging the sum with its reversal gives the exact pairing identity

$$U_N = N \sum_{j=1}^{N} \frac{1}{x_j(2N - x_j)}.$$

Since $2N \equiv 0 \pmod M$, this implies

$$\frac{U_N}{N} \equiv -\sum_{j=1}^{N} x_j^{-2} \pmod M.$$

There are 2^k odd residue classes modulo M. Our list of odd denominators visits each one exactly $N/2^k$ times. Taking modular inverses merely permutes those classes, so the sum of inverse squares equals the sum of squares modulo M:

$$\sum_{j=1}^{N} x_j^{-2} \equiv \sum_{j=1}^{N} x_j^2 = \frac{N(4N^2 - 1)}{3} \equiv N \pmod M.$$

For the final step, $(4N^2 - 1)/3$ is an odd-over-odd fraction, hence congruent to one modulo two. Multiplication by N, which contains exactly k twos, raises that congruence to modulus 2^{k+1}.

We have obtained $U_N/N \equiv -N \pmod{2^{k+1}}$. Therefore U_N/N has exactly k twos: an error divisible by 2^{k+1} cannot change that count. Multiplying by N adds another k, proving $v_2(U_N) = 2k$. Every sum and every congruence in this argument is finite.

Optional: one staircase, three exponents

Section 2 of my accompanying note, *Prime-Valuation Clocks for Truncated Zeta Sums*, gives a short arithmetic companion to Oresme's proof. The note writes $S_s(N)$ for our $Z_N(s)$. Choose k by $2^k \le N < 2^{k+1}$. This interval is a *dyadic shell*: all its indices share the same largest power of two below them.

For $s = 1$ and $s = 2$, the sums are rational and the familiar coordinate v_2 applies. At $s = 1/2$, the sums contain square roots. We use an *extended valuation* w_2: it agrees with v_2 on rationals, allows fractional values, and satisfies $w_2(xy) = w_2(x) + w_2(y)$. For example, $w_2(\sqrt{2}) = 1/2$, because squaring doubles the valuation. This extension does not assert that a sum of radicals has an ordinary rational prime vector.

The rule we need for addition is

$$w_2(x + y) \ge \min\{w_2(x), w_2(y)\},$$

with equality when the two valuations differ. In particular, **a unique summand of lowest valuation cannot be cancelled by the others**. For rational fractions this is the familiar fact that, after bringing them to a common denominator, one odd numerator plus even numerators stays odd.

Now look at the indices $1, \ldots, N$. Exactly one contains k factors of two: 2^k. Every other index contains fewer. Its reciprocal power therefore has valuation $-sk$, strictly below every other summand. The addition rule proves

$$w_2(Z_N(s)) = -sk = -s\lfloor \log_2 N \rfloor.$$

This holds for every positive *rational* exponent s, using an extension to a field containing the finitely many radicals. The result is independent of which extension above 2 is chosen. For our three exponents, the coordinates are simply $-k/2, -k, -2k$.

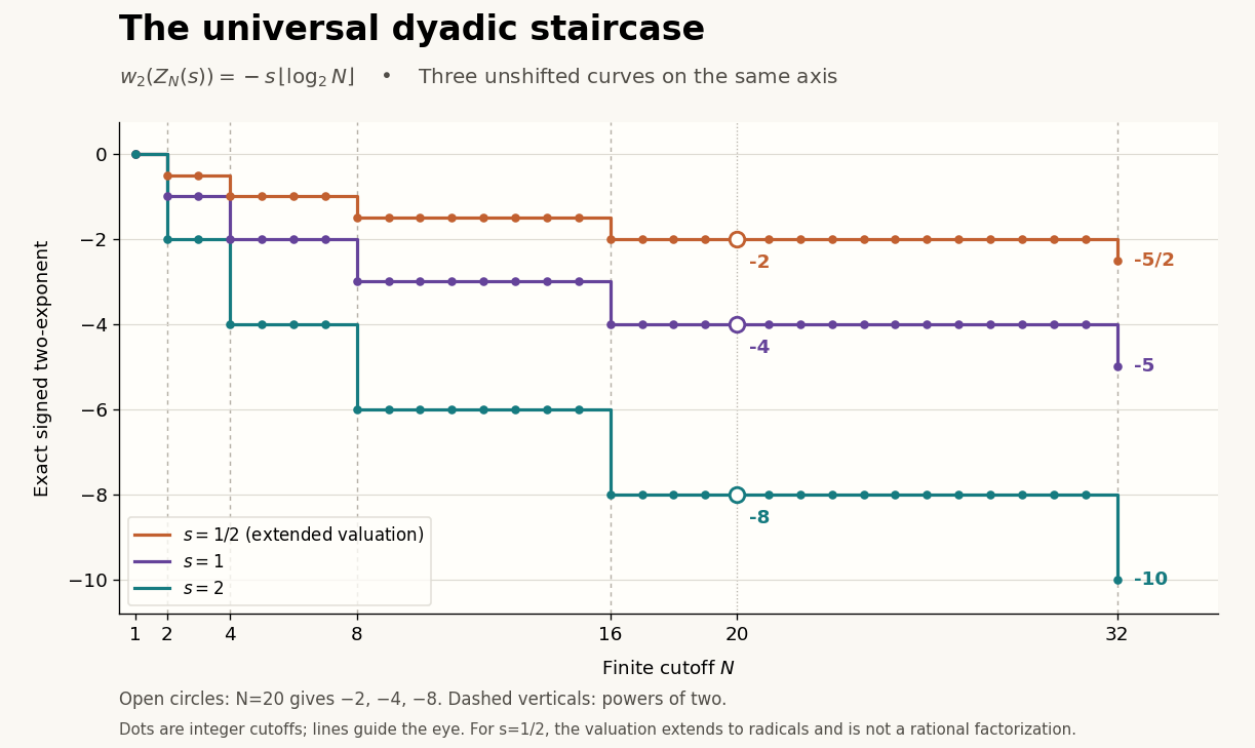

Figure 3: Three valuation staircases, shown with all curves on one common scale and no vertical offsets. Dots represent integer cutoffs N; horizontal steps are guides. At $N = 20$, the three coordinates are $-2, -4, -8$. For the square-root sum, the half-integer coordinate is the extended valuation w_2. These are divisibility coordinates, not the ordinary heights of the sums.

All three staircases descend, although only the reciprocal-square series converges. The valuation alone therefore does not decide convergence. For this particular family it tells us the shell $k = -w_2(Z_N(s))/s$; our finite bounds then translate that shell into information about ordinary size:

Exponent	Two-coordinate	Finite information when $2^k \le N < 2^{k+1}$
$s = 1/2$	$-k/2$	$2(2^{k/2} - 1) < Z_N(1/2) < 2^{(k+3)/2} - 1$
$s = 1$	$-k$	$1 + k/2 \le H_N < k + 2$
$s = 2$	$-2k$	For every finite $M > N$, $0 < Z_M(2) - Z_N(2) < 2^{-k}$

For example, at $N = 20$ the shell is $k = 4$. We obtain $6 < Z_{20}(1/2) < 8\sqrt{2} - 1$, $3 \le H_{20} < 6$, and any finite reciprocal-square tail after 20 is below $1/16$ using the shell alone. Keeping the exact endpoint improves that last bound to $1/20$. A single prime coordinate gives the scale; retaining more arithmetic data sharpens the answer.

Outlook: why harmonic numbers matter

These fractions turn up in questions as different as collecting autumn leaves, locating the boundary of convergence, and tracking prime factors. They also connect the arithmetic of finite sums to one of the central constants of analysis. Writing log for the natural logarithm,

$$H_N = \log N + \gamma_{\mathrm{EM}} + \varepsilon_N, \qquad 0 < \varepsilon_N < \frac{1}{2N}.$$

The Euler–Mascheroni constant $\gamma_{\mathrm{EM}} = 0.5772156649\ldots$ records the persistent difference between the harmonic sum and its logarithmic comparison. The explicit remainder keeps the finite cutoff in view. Jeffrey C. Lagarias's survey *Euler's constant: Euler's work and modern developments* (2013) follows this constant through number theory, probability, and its connections with the Riemann hypothesis.

An especially striking connection comes from Lagarias's *An Elementary Problem Equivalent to the Riemann Hypothesis* (2002). Define $\sigma(n)$ as the sum of the positive divisors of n; for example, $\sigma(6) = 1 + 2 + 3 + 6 = 12$. His theorem says that the Riemann hypothesis is equivalent to the inequality

$$\sigma(n) \le H_n + e^{H_n} \log H_n \qquad \text{for every integer } n \ge 1.$$

Here e is the base of the natural logarithm, distinct from γ_{EM}. A question about the zeros of the analytically continued zeta function can thus be expressed using harmonic numbers and integer divisors. The difficult part is proving the inequality for *every* integer; checking any finite range does not establish it.

That is a reason to understand harmonic numbers carefully. Their growth, exact fractions, and prime coordinates each reveal different information. Oresme's boxes and the dyadic staircase give us accessible examples of that interplay, while Lagarias's theorem shows how far the same modest-looking sums reach into mathematics.

A note from the author. I developed this material through my own exploration of the harmonic series; it may contain some original ideas. I have tried to cite related work whenever I was aware of it. If I have overlooked a prior result or connection, or stated something incorrectly, I would appreciate your comments and corrections. Please use the comments on the blog post or contact me at my official University of Jyväskylä email address: `michael.t.m.emmerich@jyu.fi`.

Materials and AI assistance. The TikZ source for the opening illustration is available for download. AI tools supported typesetting, programming, and preparation of working drafts. Every final formulation is my own. The ideas and mathematical reasoning are mine (and **not** AI generated), building on the cited literature and my notes on this topic.

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Reading further

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- Michael Emmerich, *Prime Clockwork* (2024), *On Prime Vectors and Unique Factorization of Rationals* (2025), and *Of Autumn Leaves and Coupon Collectors* (2024).
- cjackal, *2-adic valuation of odd harmonic sums*, MathOverflow, July 2016; answers by Will Sawin and Dongryul Kim.
- Tamás Lengyel, *On the 2-Adic Valuation of Differences of Harmonic Numbers*, Integers 24 (2024), A27.

Companion files

Download the TikZ illustration source (ZIP) — contains `harmonic-string-modes.tex`. Compile it with pdfLaTeX.

Download the Python sound app (ZIP) — select one or several harmonics, hear their simultaneous superposition, and compare equal amplitudes with optional reciprocal weights.

Extract the app ZIP, then run `Start Harmonic Playground.cmd` on Windows, or run `python app.py --port 0 --open-browser` in its folder. Python 3 is required; no additional packages are needed. Nothing plays until you press Play.