Prime-Valuation Clocks for Truncated Zeta Sums

Dyadic bounds, Oresme blocks, and the cases $s = \frac{1}{2}, 1, 2$

Abstract

For

$$S_s(N) = \sum_{n=1}^{N} n^{-s},$$

the valuation of an individual term is $v_p(n^{-s}) = -sv_p(n)$, after extending the valuation when s is fractional. Addition introduces cancellation, so the valuation columns of the partial sums need not be scalar multiples of one another. The dyadic column is exceptional: for every positive rational s, the index $2^{\lfloor \log_2 N \rfloor}$ supplies the unique summand of smallest 2-adic valuation, and therefore

$$w_2(S_s(N)) = -s\lfloor \log_2 N \rfloor.$$

This note develops that identity as a prime-clock description of truncated zeta sums. It explains how the dyadic valuation identifies the real dyadic shell of N and therefore yields explicit real upper and lower bounds. Oresme's grouping argument is extended to all $s > 0$, recovering the full p-series convergence/divergence trichotomy. Odd-prime columns are organized by a minutes-hand recursion $N = pm + r$. The examples $s = \frac{1}{2}, 1, 2$ illustrate fractional algebraic columns, the harmonic boundary, and rational convergent sums with faster carry freezing. Tables through $N = 30$ and a hyperreal extension are included.

1 Truncated zeta sums and signed valuation columns

For $s > 0$ define

$$S_s(N) := \sum_{n=1}^{N} \frac{1}{n^s}.$$

When s is a positive integer, every $S_s(N)$ is rational. When $s = a/b > 0$ is rational in lowest terms, the terms n^{-s} are algebraic and the partial sum lies in a finite algebraic extension of $\mathbb{Q}$.

For rational $x \neq 0$, the signed p-adic valuation is defined by

$$x = p^{v_p(x)} \frac{u}{v}, \qquad p \nmid uv.$$

Positive valuation records a numerator factor and negative valuation a denominator factor. Thus the sequence

$$N \longmapsto v_p(S_s(N))$$

may be viewed as the p-column of the partial sums.

For fractional $s = a/b$, fix an extension $w_{\mathfrak{p}}$ of v_p to a field containing the radicals, normalized by $w_{\mathfrak{p}}(p) = 1$. Then

$$w_{\mathfrak{p}}(n^{-a/b}) = -\frac{a}{b} v_p(n).$$

The value group may therefore contain fractional positions. The dyadic result below is independent of the choice of the place above 2; odd primes can exhibit place-dependent cancellation.

At the level of a single summand,

$$v_p(n^{-1/2}) = \frac{1}{2} v_p(n^{-1}), \qquad v_p(n^{-2}) = 2v_p(n^{-1}).$$

The obstruction to simply rescaling the whole valuation table is addition: several lowest-valuation summands may cancel. The prime 2 is special because the lowest summand is always unique.

The terminology and dynamical viewpoint used here are adapted from Emmerich's *Prime Clockwork* [2]. In that framework, each installed prime p carries autonomous seconds and minutes hands $R_p(n)$ and $M_p(n)$ satisfying

$$n = pM_p(n) + R_p(n), \qquad 0 \le R_p(n) < p,$$

and the minutes hand addresses an earlier local state. The present note applies that two-hand representation to the additive partial sums $S_s(N)$ rather than to the multiplicative valuation of a single integer. The elementary p-adic identities used below are not claimed to originate with the clockwork representation; what is borrowed explicitly is the state-space language and the use of the minutes hand as a recursive address.

2 The universal dyadic staircase

Theorem 2.1 (Dyadic valuation theorem)**.** *Let $s = a/b > 0$ be rational and*

$$S_s(N) = \sum_{n=1}^{N} n^{-s}.$$

For any extension w_2 of the 2-adic valuation to a field containing the summands, normalized by $w_2(2) = 1$,

$$\boxed{w_2(S_s(N)) = -s\lfloor \log_2 N \rfloor.}$$

Proof. Put $k = \lfloor \log_2 N \rfloor$, so $2^k \le N < 2^{k+1}$. Among the indices $1, \dots, N$, the integer 2^k is the unique one with $v_2(n) = k$. Hence

$$w_2((2^k)^{-s}) = -sk,$$

while every other summand has strictly larger 2-adic valuation. By the ultrametric inequality, a unique lowest-valuation term cannot be cancelled by the others. Therefore the valuation of the sum is $-sk$. □

Corollary 2.2. *For* $2^k \le N < 2^{k+1}$,

$$w_2(S_{1/2}(N)) = -\frac{k}{2}, \qquad v_2(S_1(N)) = -k, \qquad v_2(S_2(N)) = -2k.$$

Thus

$$w_2(S_{1/2}(N)) = \frac{1}{2} v_2(S_1(N)), \qquad v_2(S_2(N)) = 2v_2(S_1(N)).$$

The dyadic column is therefore a universal scale detector: changing s changes only the vertical speed of the same staircase.

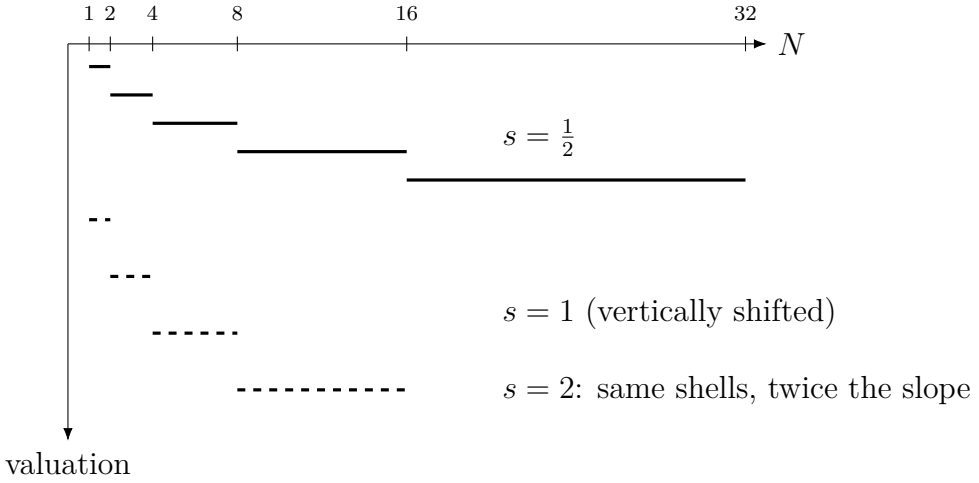

Figure 1: The same dyadic shell structure is rescaled by s.

3 Can the dyadic valuation give real upper and lower bounds?

3.1 What the valuation does – and does not – tell us

For an arbitrary rational or algebraic number, a 2-adic valuation does *not* control its ordinary real magnitude. For the family $S_s(N)$, however, Theorem 2.1 identifies something extra:

$$w_2(S_s(N)) = -sk \quad \Longleftrightarrow \quad 2^k \le N < 2^{k+1}.$$

Thus the dyadic valuation tells us exactly which real dyadic shell contains the truncation point N. Because the summands n^{-s} are positive and decreasing, standard integral comparison then converts that shell information into real upper and lower bounds.

The bridge is therefore

$$\boxed{\text{2-adic valuation} \implies \text{dyadic shell of } N \implies \text{real bounds.}}$$

The positivity and monotonicity of the zeta summands are essential; the conclusion is not a general comparison between the real and 2-adic absolute values.

3.2 Oresme's grouping for general s

The same dyadic shells already contain the classical convergence/divergence test.

Proposition 3.1 (Oresme block estimate in base p)**.** *Let $p \geq 2$ be an integer and $s > 0$. Define the kth p-power block*

$$B_{p,k}^{(s)} := \sum_{n=p^k+1}^{p^{k+1}} n^{-s}.$$

Then

$$\boxed{(p-1)p^{-s}\, p^{(1-s)k} \leq B_{p,k}^{(s)} < (p-1)p^{(1-s)k}.}$$

Proof. The block contains

$$p^{k+1} - p^k = (p-1)p^k$$

terms. Every denominator is at most p^{k+1}, so every summand is at least $p^{-s(k+1)}$; this gives the lower bound. Every denominator is strictly larger than p^k, so every summand is strictly smaller than p^{-sk}; this gives the upper bound. □

Corollary 3.2 (The p-series trichotomy from Oresme blocks)**.** *The block estimate gives the complete real behaviour of $\sum n^{-s}$:*

(i) If $0 < s < 1$, then $B_{p,k}^{(s)}$ grows like $p^{(1-s)k}$, so the series diverges.

(ii) If $s = 1$, then every block contributes at least

$$\frac{p-1}{p},$$

so the series diverges. For $p = 2$ this is Oresme's familiar lower bound $1/2$.

(iii) If $s > 1$, then the upper bounds form a convergent geometric series with ratio $p^{1-s} < 1$, so $\sum n^{-s}$ converges.

For $p = 2$ the estimate reads

$$2^{-s}2^{(1-s)k} \leq B_{2,k}^{(s)} < 2^{(1-s)k}.$$

At $s = 1$ each dyadic block contributes between $1/2$ and 1; for $s < 1$ the block contributions grow; for $s > 1$ they decay geometrically. In this sense Oresme's proof is not only a proof for the harmonic series: the same block geometry exhibits the critical exponent $s = 1$.

3.3 Integral bounds translated into the dyadic clock

For a positive decreasing function x^{-s},

$$\int_1^{N+1} x^{-s}\, dx \leq S_s(N) \leq 1 + \int_1^N x^{-s}\, dx.$$

For $s > 1$ it is more useful to apply the integral test to the tail

$$R_s(N) := \zeta(s) - S_s(N),$$

which satisfies

$$\int_{N+1}^{\infty} x^{-s}\,dx \le R_s(N) \le \int_{N}^{\infty} x^{-s}\,dx.$$

Combining these inequalities with the shell $2^k \le N < 2^{k+1}$ gives the following proposition.

Proposition 3.3 (Real bounds from the dyadic valuation)**.** *Let*

$$v := w_2(S_s(N)) = -sk, \qquad k = -\frac{v}{s} \in \mathbb{Z}_{\ge 0}.$$

Then:

(a) $0 < s < 1$**.**

$$\boxed{\frac{2^{(1-s)k}-1}{1-s} < S_s(N) < 1 + \frac{2^{(1-s)(k+1)}-1}{1-s}.}$$

Equivalently, in terms of the valuation alone,

$$\frac{2^{-(1-s)v/s}-1}{1-s} < S_s(N) < 1 + \frac{2^{(1-s)(1-v/s)}-1}{1-s}.$$

(b) $s = 1$**.**

$$\boxed{k\log 2 < H_N < 1 + (k+1)\log 2.}$$

Since $k = -v_2(H_N)$*,*

$$\boxed{-v_2(H_N)\log 2 < H_N < 1 + \Big(1 - v_2(H_N)\Big)\log 2.}$$

(c) $s > 1$**.**

$$\boxed{\frac{2^{-(s-1)(k+1)}}{s-1} \le \zeta(s) - S_s(N) \le \frac{2^{-(s-1)k}}{s-1}.}$$

Equivalently,

$$\boxed{\frac{2^{-(s-1)}}{s-1}\,2^{(s-1)v/s} \le \zeta(s) - S_s(N) \le \frac{1}{s-1}\,2^{(s-1)v/s}.}$$

The upper and lower bounds differ by the fixed factor 2^{s-1}*.*

Proof. The integral inequalities above are standard. Insert $2^k \le N < 2^{k+1}$. For $0 < s < 1$, the function x^{1-s} is increasing; for $s > 1$, x^{1-s} is decreasing. The displayed estimates follow immediately. Finally substitute $k = -v/s$ using Theorem 2.1. □

Example 3.4 (One shell, three exponents)**.** *Take* $N = 20$*. Then*

$$16 = 2^4 \le 20 < 32 = 2^5,$$

so the dyadic clock immediately gives

$$w_2(S_{1/2}(20)) = -2, \qquad v_2(H_{20}) = -4, \qquad v_2(S_2(20)) = -8.$$

s	dyadic valuation	real quantity controlled	lower bound on shell $2^k \le N < 2^{k+1}$	upper bound
$\frac{1}{2}$	$-k/2$	$S_{1/2}(N)$	$2(2^{k/2}-1)$	$2^{(k+3)/2}-1$
1	$-k$	H_N	$k\log 2$	$1+(k+1)\log 2$
2	$-2k$	$\zeta(2)-S_2(N)$	$2^{-(k+1)}$	2^{-k}

Table 1: The same dyadic clock position supplies different real information in the three regimes $s<1$, $s=1$, and $s>1$.

Without using the exact value of N beyond its dyadic shell, Proposition 3.3 yields

$$6 < S_{1/2}(20) < 8\sqrt{2}-1 \approx 10.3137,$$

while numerically

$$S_{1/2}(20) \approx 7.59526.$$

For the harmonic sum,

$$4\log 2 < H_{20} < 1+5\log 2,$$

i.e.

$$2.77259 < H_{20} < 4.46574,$$

while

$$H_{20} \approx 3.59774.$$

Finally,

$$\frac{1}{32} \le \zeta(2)-S_2(20) \le \frac{1}{16},$$

and the actual tail is approximately

$$0.0487708.$$

Thus the three dyadic valuations $-2,-4,-8$ are simply three rescalings of the same shell information $k=4$.

k	dyadic shell	$w_2(S_{1/2})$	$v_2(H_N)$	$v_2(S_2)$	$\zeta(2)-S_2(N)$
0	1	0	0	0	between 1/2 and 1
1	2:3	$-1/2$	-1	-2	between 1/4 and 1/2
2	4:7	-1	-2	-4	between 1/8 and 1/4
3	8:15	$-3/2$	-3	-6	between 1/16 and 1/8
4	16:31	-2	-4	-8	between 1/32 and 1/16

Table 2: The valuation columns identify the dyadic shell. For $s=2$ the same shell gives an immediate tail estimate within a factor 2.

Remark 3.5. *The real bounds are intentionally coarse. The point is structural: an exact local arithmetic coordinate, $w_2(S_s(N))$, identifies the dyadic scale of the truncation and therefore controls the real size of the sum or its tail up to a constant depending only on s. Finer real estimates require more information than the single dyadic column.*

4 Odd primes and the leading packet

Let p be an odd prime and write

$$p^k \le N < p^{k+1}, \qquad q = \left\lfloor \frac{N}{p^k} \right\rfloor, \qquad 1 \le q < p.$$

The terms of smallest possible p-adic valuation are

$$(p^k)^{-s}, (2p^k)^{-s}, \dots, (qp^k)^{-s}.$$

Their common scale is p^{-sk}, so the leading packet is

$$A_{p,s}(q) := \sum_{j=1}^{q} j^{-s}.$$

Proposition 4.1 (Leading-packet criterion)**.** *With the notation above,*

$$p^{sk} S_s(N) = A_{p,s}(q) + R, \qquad w_{\mathfrak{p}}(R) > 0.$$

Consequently, if $A_{p,s}(q)$ is a unit at the chosen place $\mathfrak{p} \mid p$, then

$$\boxed{w_{\mathfrak{p}}(S_s(N)) = -sk.}$$

If the leading packet is not a unit, cancellation raises the valuation.

For integer $s = a$ this becomes a finite-field computation:

$$A_{p,a}(q) = H_q^{(a)} = \sum_{j=1}^{q} \frac{1}{j^a} \pmod{p}.$$

For $s = \frac{1}{2}$ it is an algebraic packet, and its reduction may depend on the chosen place above p.

5 The minutes-hand recursion

Write

$$n = pm + r, \qquad 0 \le r < p.$$

Following the two-hand convention of the Prime Clockwork [2], write

$$n = pm + r, \qquad m = M_p(n) = \left\lfloor \frac{n}{p} \right\rfloor, \qquad r = R_p(n) = n \bmod p.$$

Thus m is the minutes reading and r the seconds reading. The point of the notation is not merely cosmetic: the minutes hand is an address to the parent state from which the local column is recursively updated. Separating indices divisible by p gives the exact identity

$$\boxed{S_s(pm + r) = p^{-s} S_s(m) + U_{p,s}(m, r),}$$

where

$$U_{p,s}(m,r) = \sum_{\substack{1\le j\le pm+r \\ p\nmid j}} j^{-s}.$$

Every summand in $U_{p,s}$ has local valuation 0, so $w_{\mathfrak{p}}(U_{p,s}) \ge 0$.

Define

$$\ell_p(n) := \lfloor \log_p n \rfloor, \qquad X_{p,s}(n) := p^{s\ell_p(n)} S_s(n).$$

Since $\ell_p(pm+r) = \ell_p(m) + 1$ for $m \ge 1$,

$$\boxed{X_{p,s}(pm+r) = X_{p,s}(m) + p^{s\ell_p(pm+r)} U_{p,s}(m,r).}$$

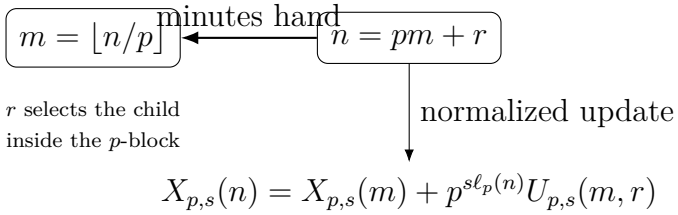

Figure 2: The minutes-hand recursion for a general exponent s.

Corollary 5.1 (Monotone subtrees)**.** *If*

$$w_{\mathfrak{p}}(S_s(m)) < s,$$

then for every $0 \le r < p$*,*

$$\boxed{w_{\mathfrak{p}}(S_s(pm+r)) = w_{\mathfrak{p}}(S_s(m)) - s.}$$

Proof. The first term $p^{-s}S_s(m)$ has valuation $w_{\mathfrak{p}}(S_s(m)) - s < 0$, while the correction has valuation at least 0. Their valuations are unequal, so no cancellation is possible. □

For integer $s = a$, the normalized state is p-integral. Modulo p^R,

$$a\ell_p(n) \ge R \quad \Longrightarrow \quad X_{p,a}(n) \equiv X_{p,a}(\lfloor n/p \rfloor) \pmod{p^R}.$$

Hence:

Proposition 5.2 (Prefix compression for integer s)**.** *Fix* $s = a \in \mathbb{Z}_{>0}$ *and* $R \ge 1$*. The state*

$$X_{p,a}(n) \pmod{p^R}$$

depends only on the first

$$\boxed{\left\lceil \frac{R}{a} \right\rceil}$$

base-p digits of n, once n has at least that many digits.

Thus $s = 2$ freezes twice as rapidly as $s = 1$ at the same ordinary p-adic precision. Fractional s has the analogous statement in the valuation filtration of the chosen local algebraic field.

6 Three cases

6.1 $s = \frac{1}{2}$: half-valuations and squarefree channels

Every positive integer has a unique representation $n = da^2$ with d squarefree. Therefore

$$\frac{1}{\sqrt{n}} = \frac{\sqrt{d}}{ad},$$

and grouping equal squarefree parts gives

$$\boxed{S_{1/2}(N) = \sum_{\substack{d \le N \\ d \text{ squarefree}}} \frac{H_{\lfloor \sqrt{N/d} \rfloor}}{d} \sqrt{d}.}$$

Thus the algebraic partial sum is assembled from ordinary rational harmonic coefficients carried by squarefree radical channels. The dyadic column has half-steps:

$$w_2(S_{1/2}(N)) = -\frac{1}{2} \lfloor \log_2 N \rfloor.$$

For the columns $p = 3$ and $p = 5$ through $N = 30$, the leading packets are units at every place above the indicated prime, so the table below is unambiguous. The first simple place-dependent example already occurs at $p = 7$, $N = 6$: the packet

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{2} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}}$$

has norm divisible by 7, and its valuation depends on the chosen place above 7. This is why the half-exponent table below stops at $p = 5$.

6.2 $s = 1$: the harmonic boundary

Here $S_1(N) = H_N$ is rational and

$$v_2(H_N) = -\lfloor \log_2 N \rfloor.$$

This is the critical Oresme case: every dyadic block contributes at least $1/2$, so the real series diverges only logarithmically. Odd-prime columns show immediate cancellations; for example $H_2 = 3/2$ has $v_3(H_2) = 1$.

6.3 $s = 2$: rational convergence and faster carry freezing

Every row

$$S_2(N) = \sum_{n \le N} \frac{1}{n^2}$$

is rational and

$$v_2(S_2(N)) = -2 \lfloor \log_2 N \rfloor.$$

At precision p^R, only $\lceil R/2 \rceil$ leading base-p digits are needed for the normalized carry. Odd-prime cancellation differs from the harmonic case. At $p = 5$,

$$1 + \frac{1}{2^2} = 1 + \frac{1}{4} \equiv 0 \pmod 5,$$

so the leading level cancels in the block with leading digit 2. More generally, for every prime $p > 3$,

$$\sum_{j=1}^{p-1} j^{-2} \equiv 0 \pmod p, \qquad \sum_{j=1}^{(p-1)/2} j^{-2} \equiv 0 \pmod p.$$

7 Valuation tables through $N = 30$

The complete numerical tables are collected in Appendices A–C, one table per page. Keeping them out of the main text makes the structural comparison easier to read while preserving the full data set.

8 The common skeleton and the exponent-dependent carry

For rational $s > 0$ it is useful to write schematically

$$\boxed{w_{\mathfrak{p}}(S_s(N)) = -s\ell_p(N) + D_{p,s}(N),}$$

where $D_{p,s}(N)$ records the cancellation above the prime-scale baseline. At $p = 2$,

$$D_{2,s}(N) = 0$$

identically. Thus the dyadic column is an exact linear probe of the exponent s.

At odd primes, the first correction is governed by the leading packet

$$A_{p,s}(q) = \sum_{j=1}^{q} j^{-s}.$$

For $s = 1$ it is an ordinary harmonic packet, for $s = 2$ a second-order harmonic packet, and for $s = \frac{1}{2}$ an algebraic packet whose reduction can branch with the choice of place.

s	row arithmetic	dyadic step	local carry
$\frac{1}{2}$	multiquadratic algebraic	$-\frac{1}{2}$	fractional / place-dependent
1	rational	-1	harmonic
2	rational	-2	faster freezing

9 Hyperreal extension

Let $K \in {}^*\mathbb{N}$ be an infinite hyperinteger. By transfer,

$$S_s(K) = \sum_{n=1}^{K} n^{-s}$$

is a hyperfinite sum and Theorem 2.1 becomes

$$\boxed{{}^*w_2(S_s(K)) = -s\lfloor \log_2 K \rfloor.}$$

Thus an infinite index produces an infinitely negative dyadic valuation even when the real hypervalue is finite.

If $s > 1$, then $S_s(K)$ is finite and

$$\boxed{\mathrm{st}(S_s(K)) = \zeta(s).}$$

For $s = 2$,

$$\mathrm{st}(S_2(K)) = \frac{\pi^2}{6}, \qquad {}^*v_2(S_2(K)) = -2\lfloor \log_2 K \rfloor.$$

The same dyadic valuation also gives a hyperreal tail estimate:

$$\frac{1}{2}\, 2^{{}^*v_2(S_2(K))/2} \le \zeta(2) - S_2(K) \le 2^{{}^*v_2(S_2(K))/2},$$

where both bounds are positive infinitesimals.

At $s = 1$, the hyperfinite harmonic sum H_K is infinite and has no finite real standard part. This is the logarithmic boundary case.

For $0 < s < 1$,

$$S_s(N) = \frac{N^{1-s}}{1-s} + \zeta(s) + o(1),$$

so

$$\boxed{\mathrm{st}\left(S_s(K) - \frac{K^{1-s}}{1-s}\right) = \zeta(s).}$$

At the central point,

$$\boxed{\mathrm{st}\Big(S_{1/2}(K) - 2\sqrt{K}\Big) = \zeta(1/2).}$$

Since

$$w_2(2\sqrt{K}) = 1 + \tfrac{1}{2} v_2(K) \ge 1$$

while $w_2(S_{1/2}(K))$ is infinitely negative, subtraction of the real counterterm does not change the dyadic valuation:

$$\boxed{{}^*w_2\Big(S_{1/2}(K) - 2\sqrt{K}\Big) = -\frac{1}{2}\lfloor \log_2 K \rfloor.}$$

Thus the finite real standard part forgets a dyadic coordinate that remains infinitely deep in the exact hyperalgebraic representative.

10 Perspective

The comparison $s = \frac{1}{2}, 1, 2$ separates two mechanisms. The universal scale skeleton

$$-s\lfloor \log_p N \rfloor$$

is exact at $p = 2$. Odd primes add exponent-dependent cancellation. The minutes-hand recursion turns these cancellations into local dynamical states, while the dyadic theorem converts one exact arithmetic column into a real scale coordinate for the truncation.

Oresme's grouping gives a complementary real interpretation. Crossing a new dyadic shell lowers the dyadic valuation by exactly s, while the real contribution of that shell is of size

$$2^{(1-s)k}.$$

At $s < 1$ those block contributions grow, at $s = 1$ they remain of constant order, and at $s > 1$ they decay geometrically. The valuation staircase and the real p-series trichotomy are therefore two views of the same prime-power scale decomposition.

This suggests studying

$$(p, s, N) \longmapsto D_{p,s}(N)$$

as a cancellation spectrum for truncated zeta sums: the dyadic column measures the exponent directly, while the odd columns record how addition modifies the universal prime-scale skeleton.

A Half-valuation table: $s = \frac{1}{2}$

The table lists the selected local valuation columns through $N = 30$. It is placed on its own page so that the row structure and dyadic staircases remain visually legible.

N	w_2	w_3	w_5
1	0	0	0
2	$-\frac{1}{2}$	0	0
3	$-\frac{1}{2}$	$-\frac{1}{2}$	0
4	-1	$-\frac{1}{2}$	0
5	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
6	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
7	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
8	$-\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
9	$-\frac{3}{2}$	-1	$-\frac{1}{2}$
10	$-\frac{3}{2}$	-1	$-\frac{1}{2}$
11	$-\frac{3}{2}$	-1	$-\frac{1}{2}$
12	$-\frac{3}{2}$	-1	$-\frac{1}{2}$
13	$-\frac{3}{2}$	-1	$-\frac{1}{2}$
14	$-\frac{3}{2}$	-1	$-\frac{1}{2}$
15	$-\frac{3}{2}$	-1	$-\frac{1}{2}$
16	-2	-1	$-\frac{1}{2}$
17	-2	-1	$-\frac{1}{2}$
18	-2	-1	$-\frac{1}{2}$
19	-2	-1	$-\frac{1}{2}$
20	-2	-1	$-\frac{1}{2}$
21	-2	-1	$-\frac{1}{2}$
22	-2	-1	$-\frac{1}{2}$
23	-2	-1	$-\frac{1}{2}$
24	-2	-1	$-\frac{1}{2}$
25	-2	-1	-1
26	-2	-1	-1
27	-2	$-\frac{3}{2}$	-1
28	-2	$-\frac{3}{2}$	-1
29	-2	$-\frac{3}{2}$	-1
30	-2	$-\frac{3}{2}$	-1

B Harmonic valuation table: $s = 1$

The table lists the selected local valuation columns through $N = 30$. It is placed on its own page so that the row structure and dyadic staircases remain visually legible.

N	v_2	v_3	v_5	v_7
1	0	0	0	0
2	-1	1	0	0
3	-1	-1	0	0
4	-2	-1	2	0
5	-2	-1	-1	0
6	-2	0	-1	2
7	-2	1	-1	-1
8	-3	0	-1	-1
9	-3	-2	-1	-1
10	-3	-2	-1	-1
11	-3	-2	-1	-1
12	-3	-2	-1	-1
13	-3	-2	-1	-1
14	-3	-2	-1	-1
15	-3	-2	-1	-1
16	-4	-2	-1	-1
17	-4	-2	-1	-1
18	-4	-1	-1	-1
19	-4	-1	-1	-1
20	-4	-1	1	-1
21	-4	0	0	-1
22	-4	1	0	-1
23	-4	0	0	-1
24	-4	-1	1	-1
25	-4	-1	-2	-1
26	-4	-1	-2	-1
27	-4	-3	-2	-1
28	-4	-3	-2	-1
29	-4	-3	-2	-1
30	-4	-3	-2	-1

C Second-order valuation table: $s = 2$

The table lists the selected local valuation columns through $N = 30$. It is placed on its own page so that the row structure and dyadic staircases remain visually legible.

N	v_2	v_3	v_5	v_7
1	0	0	0	0
2	-2	0	1	0
3	-2	-2	0	2
4	-4	-2	1	0
5	-4	-2	-2	0
6	-4	-2	-2	1
7	-4	-2	-2	-2
8	-6	-2	-2	-2
9	-6	-4	-2	-2
10	-6	-4	-1	-2
11	-6	-4	-1	-2
12	-6	-4	-1	-2
13	-6	-4	-1	-2
14	-6	-4	-1	-2
15	-6	-4	-2	-2
16	-8	-4	-2	-2
17	-8	-4	-2	-2
18	-8	-4	-2	-2
19	-8	-4	-2	-2
20	-8	-4	-1	-2
21	-8	-4	-1	0
22	-8	-4	-1	0
23	-8	-4	-1	0
24	-8	-4	-1	0
25	-8	-4	-4	0
26	-8	-4	-4	1
27	-8	-6	-4	0
28	-8	-6	-4	-2
29	-8	-6	-4	-2
30	-8	-6	-4	-2

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