

# Shaping Expected Hypervolume and $R_2$ Improvement based on Decision Maker Preferences

Exact Integration, Pareto Compatibility, and Variance Monotonicity

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*Preference-Shaped Expected Hypervolume and  $R_2$  Improvement: Exact Computation and Monotonicity*

**arXiv:2605.28746 [math.OC], 2026**

## In this talk

- main geometry and results,
- computation principles,
- theorems and propositions without long proofs.

## In the report

- exact statements and assumptions,
- complete proofs and appendices,
- technical derivations for shaped EHVI and ER2I.

Use this as the proof companion to the presentation. Find link to slides at:  
[www.emmerix.net](http://www.emmerix.net)

## Guiding Question

How can preferences be built into expected multi-objective improvement acquisition functions without losing exact computability, Pareto compliance, and active learning reward?

## Hypervolume / EHVI

- dominated-volume geometry,
- preference shaping by densities, cones, or truncation,
- Convenient: Deng et al. (AAAI 2025) representation transfers EHVI to particular HVI.

## $R_2$ / ER2I

- scalarization-envelope geometry,
- discrete weights versus exact integration,
- Tchebycheff shadows give the right integral domain.

**Three lenses:** exact computation, Pareto compatibility, mean value and variance monotonicity.

**Preference types:** Utopian/dystopian reference, weighting with positive kernels, cone-order, conditioned samples

# Related work: standing on shoulders of giants

## EHVI lineage

- Emmerich (2005); Emmerich, Giannakoglou, Naujoks (2006)
- Wagner, Emmerich, Deutz, Ponweiser (2010)
- Emmerich, Deutz, Klinkenberg (2011)
- Deng, Sun, Zhang, Li (2025)

## Preference-shaped HV

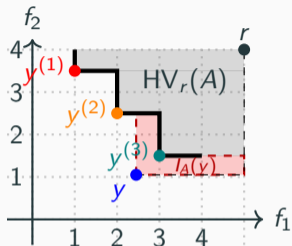
- Auger, Bader, Brockhoff, Zitzler (2009)
- Wagner and Trautmann (2010)
- Emmerich, Deutz, Yevseyeva (2014)
- Emmerich, Deutz, Kruisselbrink, Shukla (2013)
- Yang, Li, Deutz, Bäck, Emmerich (2016)
- Liang, Shavazipour, Saini, Emmerich (2026)

## $R_2$ and ER2I

- Hansen and Jaszkievicz (1994)
- Brockhoff, Wagner, Trautmann (2012)
- Deutz, Emmerich, Yang (2019)
- Schäpermeier and Kerschke (2024)
- Jaszkievicz and Zielniewicz (2024)

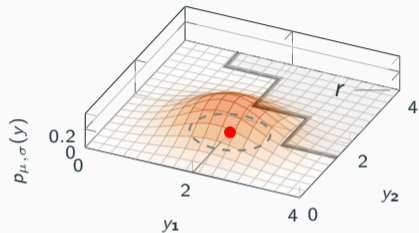
**Positioning.** The report connects these strands by testing which preference transformations preserve exact computation, Pareto compatibility, and variance monotonicity.

# Expected Hypervolume Improvement: geometry and predictive expectation



Hypervolume Improvement  $I_A(y)$

$$\begin{aligned} I_A(y) &= \text{HVI}_r(\{y\}, A) \\ &= \text{HV}_r(A \cup \{y\}) - \text{HV}_r(A). \end{aligned}$$



Probabilistic part

$$\begin{aligned} \text{EHVI}_A(\mu, \sigma) &= \mathbb{E}[I_A(Y)], \\ Y &= \mu + \sigma \odot Z, \\ Z &\sim N(0, I_m). \end{aligned}$$

EHVI averages the hypervolume improvement over the predictive distribution.

M. T. M. Emmerich, *Single- and Multiobjective Evolutionary Design Optimization Assisted by Gaussian Random Field Metamodels*, doctoral dissertation, TU Dortmund, 2005.

M. T. M. Emmerich, K. C. Giannakoglou, and B. Naujoks, *Single- and multiobjective evolutionary optimization assisted by Gaussian random field metamodels*, IEEE TEC, 2006.

## Theorem (Deng–Sun–Zhang–Li, 2025)

For independent Gaussian predictive coordinates, EHVI can be written as a particular ordinary hypervolume improvement after applying coordinate-wise expected-improvement transformations.

For maximization and reference 0:

$$\begin{aligned}\tilde{r}_j &= \text{El}(0; \mu_j, \sigma_j^2), \\ \tilde{a}_j^{(i)} &= \tilde{r}_j - \text{El}(a_j^{(i)}; \mu_j, \sigma_j^2), \\ \text{EHVI}_A(\mu, \sigma) &= \text{HVI}(\{\tilde{r}\}, \tilde{A}).\end{aligned}$$

**Consequence:** exact EHVI computation and structural questions can be transferred to ordinary HVI geometry.

J. Deng, J. Sun, Q. Zhang, and H. Li, *Expected hypervolume improvement is a particular hypervolume improvement*, AAAI, 2025.

# Three guiding modes

## 1. Computation

- exact formula or tractable quadrature?
- ordinary HVI, scalar EI, or new integral?

## 2. Pareto compatibility

- better predictive mean should not be rewarded,
- preference shaping should comply with Pareto dominance.

## 3. Variance monotonicity

- larger predictive uncertainty should not reduce exploration value,
- $\Rightarrow$  reward active learning/sampling in underexplored regions.

### Proposition (baseline)

For independent Gaussian predictive coordinates, canonical EHVI is mean-monotone and variance-monotone in arbitrary dimension.

**T. Wagner, M. T. M. Emmerich, A. H. Deutz, and W. Ponweiser**, *On expected-improvement criteria for model-based multi-objective optimization*, PPSN XI, 2010.

**M. T. M. Emmerich, A. H. Deutz, and J. W. Klinkenberg**, *Hypervolume-based expected improvement: monotonicity properties and exact computation*, IEEE CEC, 2011.

**J. Deng, J. Sun, Q. Zhang, and H. Li**, *Expected hypervolume improvement is a particular hypervolume improvement*, AAAI, 2025.

# Product-density EHVI: preferences as coordinate transformations

**Preference model.** Replace Lebesgue volume by an (almost everywhere) positive product kernel

$$K(y) = \prod_{j=1}^m k_j(y_j), \quad T_j(y_j) = \int_{-\infty}^{y_j} k_j(s) ds.$$

A desirability function  $D_j$  is one concrete choice:  $T_j = D_j$  and  $k_j = D_j'$ .  $T_j \sim$  weight; weighted HV.

**Note:**  $T_j$  can be negative, but  $k_j$  not

**Interpretation.** The local slope of  $D_j$  says where an additional objective improvement is most valuable.

## Proposition (positive product density)

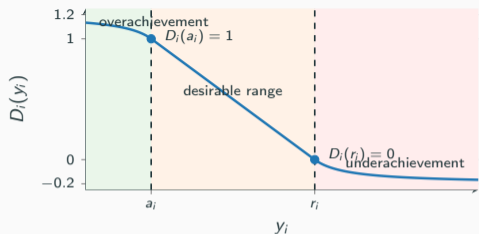
Weighted hypervolume improvement equals ordinary HVI after the monotone coordinate map  $T = (T_1, \dots, T_m)$ .

- **Computation:** inherited from HVI/EHVI.
- **Pareto compatibility:** preserved by strict monotonicity.
- **Variance monotonicity:** preserved for Gaussian models in transformed coordinates.

A. Auger, J. Bader, D. Brockhoff, and E. Zitzler, *Articulating user preferences in many-objective problems by sampling the weighted hypervolume*, GECCO, 2009.

M. T. M. Emmerich, A. H. Deutz, and I. Yevseyeva, *On reference point free weighted hypervolume indicators based on desirability functions and their probabilistic interpretation*, Procedia Technology, 2014.

# Example product transformation: D-PHI desirability functions



**Product-kernel view.** The marginal product-kernel factors are induced by the local slopes:

$$k_i(y_i) \propto |D'_i(y_i)|.$$

**D-PHI idea.** Aspiration and reservation levels define

$$T_i(y_i) = D_i(y_i), \quad D_i(a_i) = 1, \quad D_i(r_i) = 0,$$

where  $a_i$  is full satisfaction and  $r_i$  is dissatisfaction for minimization.

## Definition (D-PHI)

For a set  $P$  and its image  $T(P)$  in desirability space,

$$D\text{-PHI}(P) = \text{HV}(T(P), d_{\text{worst}}).$$

Hence aspiration–reservation levels shape the local density of hypervolume improvement.

**M. Liang, B. Shavazipour, B. Saini, and M. Emmerich, *D-PHI: Desirability-Based Hypervolume Indicator for Interactive Multiobjective Optimization Using Aspiration and Reservation Levels as Preferences*, ACM TELO, 2026.**

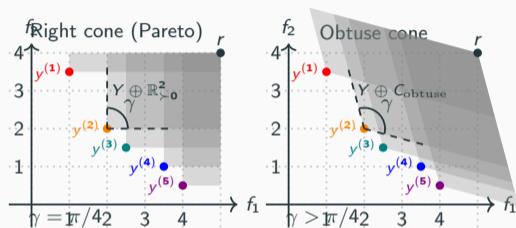
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**Ogryczak, W., & Lahoda, S. (1992).** Aspiration/reservation-based decision support—a step beyond goal programming. *Journal of Multi-Criteria Decision Analysis*, 1(2), 101–117.

**Wagner, T., & Trautmann, H. (2010).** Integration of preferences in hypervolume-based multiobjective evolutionary algorithms by means of desirability functions. *IEEE Transactions on Evolutionary Computation*, 14(5), 688–701.

# Cone-based hypervolume: preferences as trade-off constraints

The cone angle encodes preference for trade-off directions; in particular, cone orders can model maximum trade-off constraints. Cone order:  $y \prec y' \Leftrightarrow y' \in y \oplus C$ . (Minkowsky sum  $\oplus$ ,  $C$  cone)



## Proposition

For a simplicial cone  $C$  with  $L = C^{-1}$ ,

$$\text{CHV}_{C,r}(A) = |\det C| \text{HV}_{Lr}(LA).$$

Hence cone-based EHVI is ordinary EHVI in cone coordinates, up to a positive factor.

M. T. M. Emmerich, A. H. Deutz, J. Krusselbrink, and P. K. Shukla, *Cone-based hypervolume indicators: construction, properties, and efficient computation*, EMO, LNCS 7811, 2013.

# Truncated EHVI: preferences as a region of interest

**Preference model.** The decision maker specifies a preferred objective-space box  $[a, b]$  and the Gaussian density is conditioned on  $Y \in [a, b]$ :

$$\text{TEHVI}_A(\mu, \sigma; a, b) = \int_{[a, b]} I_A(y) p_{\mu, \sigma}^{[a, b]}(y) \, dy.$$

## Proposition (fixed truncation box)

For independent truncated coordinates, a Deng et al. type representation exists and mean monotonicity holds, but variance monotonicity can fail.

## Takeaway

Truncation is useful for region-of-interest search, but it changes the exploration principle.

K. Yang, L. Li, A. H. Deutz, T. Bäck, and M. T. M. Emmerich, *Preference-based multiobjective optimization using truncated expected hypervolume improvement*, ICNC-FSKD, 2016.

# $R_2$ : the ideal-point counterpart of hypervolume

## Hypervolume

- Uses a dystopian reference point  $r$ .
- Measures dominated volume.
- Improvement adds newly dominated volume.

## $R_2$ indicator

- Uses a utopian / ideal point  $z^+$ .
- Uses scalarizing functions and weights.
- Improvement decreases the best scalarizing loss.

**Discrete**  $R_2$  was introduced by Hansen and Jaszkiwicz; its properties were studied by Brockhoff, Wagner, and Trautmann. Weakly Pareto compliant.

**Exact** (Integrated)  $R_2$  Indicator (Schäpermeier+ 2024, Jaszkiwicz+ 2024); Pareto compliant)  
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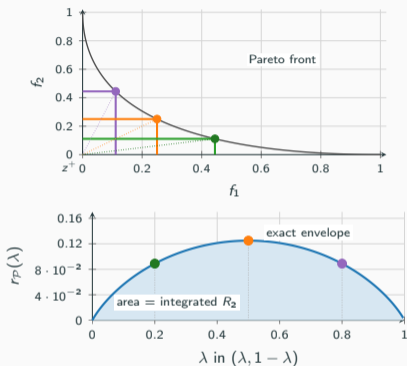
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L. Schäpermeier and P. Kerschke, *Reinvestigating the  $R_2$  indicator: achieving Pareto compliance by integration*, PPSN XVIII, LNCS 15151, 2024.

A. Jaszkiwicz and P. Zielniewicz, *Exact calculation and properties of the  $R_2$  multiobjective quality indicator*, IEEE TEC, 2024.

# Understanding $R_2$ : objective space and weight space



**Objective-space view.** Utopian point  $z^+$ , weights  $\lambda \in [0, 1]$ ,

$$g_\lambda(a; z^+) = \max\{\lambda(a_1 - z_1^+), (1 - \lambda)(a_2 - z_2^+)\}.$$

**Weight-space view.** Discrete  $R_2$  samples finitely many weights, while integrated  $R_2$  uses the exact envelope

$$r_{\mathcal{P}}(\lambda) = \min_{a \in \mathcal{P}} g_\lambda(a; z^+).$$

**Definition (Integrated  $R_2$  in 2-D)**

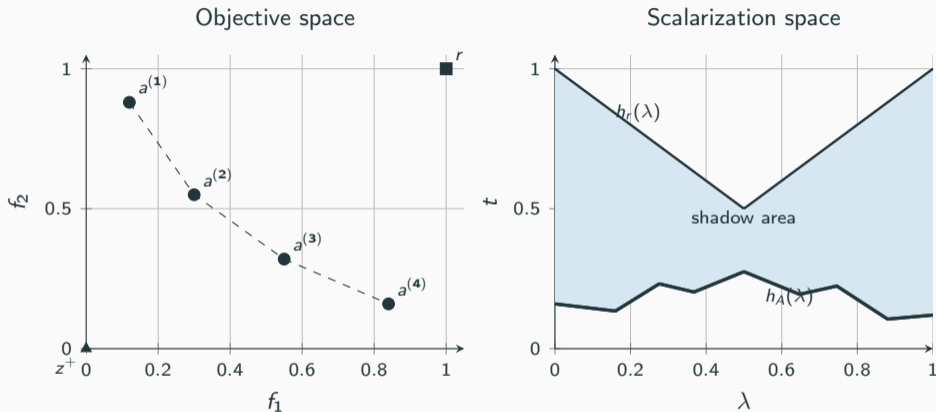
$$R_2(\mathcal{P}) = \int_0^1 r_{\mathcal{P}}(\lambda) d\lambda.$$

M. P. Hansen and A. Jasziewicz, *Evaluating the quality of approximations to the non-dominated set*, DTU IMM technical report, 1994.

L. Schäpermeier and P. Kerschke, *Reinvestigating the  $R_2$  indicator: achieving Pareto compliance by integration*, PPSN XVIII, LNCS 15151, 2024.

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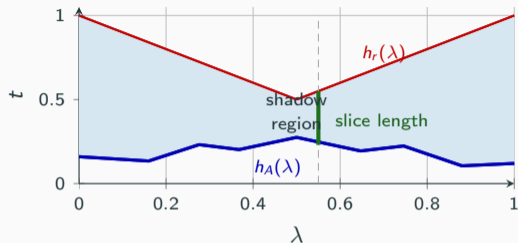
# Tchebycheff shadows: $R_2$ cannot be represented as weighted HV



## Theorem (Tchebycheff shadow representation)

Exact integral  $R_2$  improvement equals the scalarization-space volume between  $h_A(\lambda)$  and  $h_r(\lambda)$ , and cannot be obtained by any ordinary objective-space weighted hypervolume.

# Layer-cake step: from shadow slices to volume



## Statement (Layer-cake interpretation)

The improvement integral is the weighted volume of the Tchebycheff shadow

$$\mathcal{H}(A; r) = \{(\lambda, t) : h_A(\lambda) \leq t \leq h_r(\lambda)\}.$$

$$I_{R^2}(A; r) = \int_{\mathcal{H}(A; r)} \rho(\lambda) dt d\lambda.$$

## Lemma (layer-cake identity)

For each fixed weight  $\lambda$ ,

$$(h_r(\lambda) - h_A(\lambda))_+ = \int_0^{h_r(\lambda)} \mathbf{1}\{h_A(\lambda) \leq t\} dt.$$

Thus the integrand is the length of a vertical slice. Integrating these slice lengths over the weight simplex gives

$$I_{R^2}(A; r) = \int_{\Delta^{m-1}} \int \mathbf{1}\{h_A(\lambda) \leq t \leq h_r(\lambda)\} \rho(\lambda) dt d\lambda.$$

## Lemma (Fubini's theorem)

Since the integrand is nonnegative, the order of integration may be exchanged.

# Towards expected improvement of the $R_2$ indicator: ER2I

For a finite weight set  $\Lambda$ , define a scalarizing loss

$$G_\lambda(A) = \min_{a \in A} g_\lambda(a; z^+).$$

A candidate random vector  $Y$  improves the set for weight  $\lambda$  by

$$\Delta_\lambda(Y; A) = (G_\lambda(A) - \min\{G_\lambda(A), g_\lambda(Y; z^+)\})_+.$$

## Definition (Discrete ER2I)

$$\text{ER2I}_\Lambda(Y; A) = \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \mathbb{E}[\Delta_\lambda(Y; A)].$$

- Exact for finite  $\Lambda$ : finitely many scalar Gaussian EI-type integrals.
- Background and computation: Deutz–Emmerich–Yang.

A. H. Deutz, M. T. M. Emmerich, and K. Yang, *The expected  $R_2$ -indicator improvement for multi-objective Bayesian optimization*, EMO, LNCS 11411, 2019.

# Computing discrete and exact ER2I

Discrete weights: scalar GP/EI per  $\lambda$

$$\text{ER2I}_\Lambda(x) = \frac{1}{K} \sum_{k=1}^K \text{EI}\left(h_A(\lambda^{(k)}); m_k(x), s_k(x)\right).$$

- one scalar achievement model per sampled weight,
- closed-form scalar EI at every node,
- finite sum after updating  $h_A(\lambda^{(k)})$ .

Exact  $R_2$ : Tchebycheff-shadow quadrature

$$\text{ER2I}_{\text{int}}(x) \approx \sum_{\ell=1}^L w_\ell \rho(\lambda^{(\ell)}) \text{EI}\left(h_A(\lambda^{(\ell)}); m_x(\lambda^{(\ell)}), s_x(\lambda^{(\ell)})\right).$$

- integrate over the simplex  $\Delta^{m-1}$ ,
- in two objectives: one-dimensional quadrature,
- objective-GP models require an inner normal-CDF integral.

## Main computational message

Discrete ER2I is a finite EI sum; exact integral ER2I is quadrature over the Tchebycheff shadow.

# Variance monotonicity for ER2I

## Theorem (ER2I variance monotonicity)

**Achievement-space Gaussian model.** Finite-weight ER2I and exact/quadrature ER2I are nondecreasing in every achievement standard deviation  $s_x(\lambda)$ .

**Objective-space Gaussian model.** At weight  $\lambda$ , monotonicity in objective standard deviation  $\sigma_j$  is guaranteed under

$$h_A(\lambda) \leq \alpha_j(\lambda), \quad \alpha_j(\lambda) = \lambda_j(\mu_j(x) - z_j^+).$$

Without this threshold condition, monotonicity can fail.

## Practical interpretation

Achievement-space ER2I gives scalar EI nodes. Objective-space ER2I is more faithful to the joint objective model, but the Tchebycheff maximum is not Gaussian.

# Comparison by guiding mode

Criterion	Computation	Pareto compatibility	Variance monotonicity
Canonical EHVI	exact via HVI; Deng representation	yes	yes, all dimensions
Product-density EHVI	exact after monotone coordinate map	yes	yes in transformed coordinates
Cone EHVI	exact after cone-coordinate map	cone-order compatible	yes in cone coordinates
Truncated EHVI	exact/truncated variant	Gaussian fixed ROI	can fail
Discrete ER2I	finite scalar EI sum	weight-set dependent	yes for achievement GPs; conditional for objective GPs
Exact ER2I	Tchebycheff-shadow quadrature on $\Delta^{m-1}$	yes for integrated $R_2$	yes for achievement GPs; conditional for objective GPs

**Message.** EHVI lives in objective-space dominated volume; exact  $R_2$  improvement lives in scalarization-space shadow volume.

# Takeaways and next steps

## Main results

- Deng representation clarifies EHVI computation and monotonicity.
- Product densities and cones preserve HVI-type structure.
- Truncation focuses search but can break variance monotonicity.
- Tchebycheff shadows give the right exact-integral view of  $R_2$  improvement.

## Next steps

- robust finite-weight and adaptive-quadrature ER2I implementations,
- diagnostics for the objective-GP variance condition,
- comparisons of shaped EHVI, achievement-space ER2I, and objective-space ER2I.

## Open-source direction

Python implementations for exact EHVI and ER2I to be released through the DEMO/DESDEO ecosystem.

## Next step: Empirical evaluation

**Teaser: preference-guided multiobjective Bayesian optimization**  
M. Liang, J. Branke, B. Saini, K. Miettinen, M. T. M. Emmerich

*Preference-Guided Multiobjective Bayesian Optimization with Aspiration and Reservation Levels*

Accepted for **PPSN 2026**, Trento, Italy.

### Idea

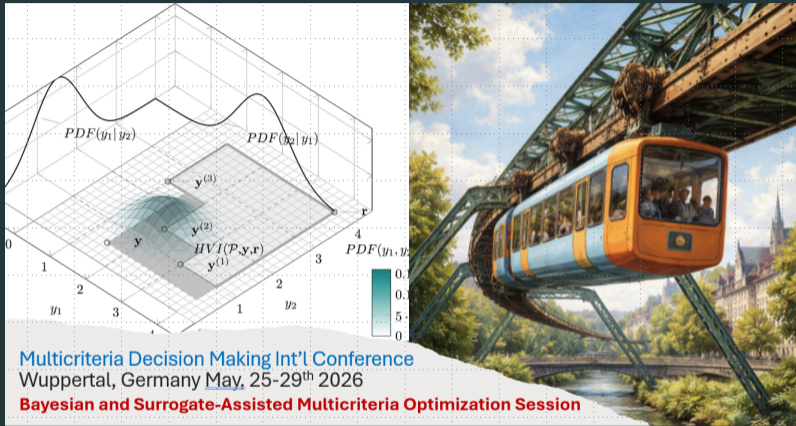
- Make Bayesian optimization preference-aware from the start.
- Use **aspiration levels** to focus search on fully satisficing objective function levels.
- Use **reservation levels** to avoid practically unacceptable objective levels.

### Connection to this talk

- Preference-shaped improvement indicators provide a principled acquisition signal.
- Theory: desirability-based weighted HV kernels.
- Next: empirical comparison against standard EHVI /  $R_2$ -based BO.

From exact preference-shaped indicators to practical preference-guided BO.

Thank you.



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## Appendix: achievement-space variance proof

For fixed  $c$ ,  $\mu$ , and  $s > 0$ ,

$$\text{EI}(c; \mu, s) = (c - \mu)\Phi\left(\frac{c - \mu}{s}\right) + s\varphi\left(\frac{c - \mu}{s}\right).$$

Direct differentiation gives

$$\frac{\partial}{\partial s} \text{EI}(c; \mu, s) = \varphi\left(\frac{c - \mu}{s}\right) \geq 0.$$

Therefore each node contribution is nondecreasing in the achievement standard deviation. The finite-weight criterion and the quadrature/integrated criterion are nonnegative weighted sums or integrals of such node contributions, hence preserve the monotonicity.

## Appendix: objective-space Gaussian derivative

Fix  $\lambda$  in the relative interior of the simplex and put

$$X_i = \lambda_i(Y_i - z_i^+), \quad \alpha_i = \lambda_i(\mu_i - z_i^+), \quad \beta_i = \lambda_i\sigma_i, \quad c = h_A(\lambda).$$

The exact node-wise objective-GP improvement is

$$I_\lambda = \int_{-\infty}^c \prod_{i=1}^m \Phi\left(\frac{t - \alpha_i}{\beta_i}\right) dt.$$

Differentiating with respect to  $\beta_j$  gives

$$\frac{\partial I_\lambda}{\partial \beta_j} = \int_{-\infty}^c \prod_{i \neq j} \Phi\left(\frac{t - \alpha_i}{\beta_i}\right) \left[ -\frac{t - \alpha_j}{\beta_j^2} \varphi\left(\frac{t - \alpha_j}{\beta_j}\right) \right] dt.$$

If  $c \leq \alpha_j$ , then  $t \leq c \leq \alpha_j$  throughout the integral, so the bracketed factor is nonnegative. Finite sums, quadrature sums, and integrals preserve this node-wise monotonicity when the condition holds at all relevant nodes, respectively almost everywhere.

## Appendix: objective-space counterexample

When  $c > \alpha_j$ , the derivative integrand changes sign, so no unconditional objective-GP variance theorem follows.

A simple node-wise counterexample uses two scaled Gaussian components with

$$c = 1, \quad \alpha_1 = \alpha_2 = 0, \quad \beta_2 = 1.$$




Then




$$I(\beta_1) = \int_{-\infty}^1 \Phi\left(\frac{t}{\beta_1}\right) \Phi(t) \, dt,$$



and numerical evaluation gives




$$I(0.1) \approx 0.6824, \quad I(0.2) \approx 0.6765.$$

Thus increasing one component standard deviation can decrease an objective-GP ER2I contribution. This applies directly to a finite-weight indicator and to any quadrature rule with positive weight at such a node; by continuity it also affects continuous densities concentrated near that node.





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