Book Overhang with Four Blocks: A Clear Mechanical Model and Two Worked Examples

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Abstract

How far can a stack of identical "books" (idealized as rigid, frictionless blocks) extend beyond the edge of a table without toppling? This note rewrites the mechanical model used in the classic overhang literature and works out two small examples in full coordinate detail: a three-block configuration with overhang 1, and the four-block "Ainley" configuration with overhang $\frac{15-4\sqrt{2}}{8} \approx 1.16789$ as reproduced by Paterson et al. (4). Throughout we use a one-dimensional, vertical-force-only model and treat balance via force propagation and centers of gravity. A companion interactive implementation that uses the same endpoint-reduction viewpoint is available online (2).

1 Coordinate convention and the vertical-force model

We normalize every block to have length 1 and weight 1. The table occupies the half-plane *to the left* of its edge:

$$\text{table support region } = \{(x, y) : x \leq 0\}.$$

We draw the table edge as the vertical line $x = 0$.

Meaning of the variable x. The variable x is *the horizontal coordinate along the table edge line*, measured in *block lengths*. Positive x means "to the right" (overhanging side), negative x means "to the left" (supported side). Each block i is represented by a closed interval on the x-axis,

$$B_i = [x_i, x_i + 1] \subset \mathbb{R},$$

where x_i is the *left endpoint* of the block. Its center of gravity is at

$$\text{cg}(B_i) = x_i + \tfrac{1}{2}.$$

Only *vertical* forces are allowed (no shear, no torque couples, no friction). A stacked configuration is *balanced* if there exists a set of nonnegative vertical reaction forces at contact regions that satisfies: (i) force equilibrium, and (ii) moment equilibrium for each block. The model and its "force propagation" viewpoint are standard in the overhang literature; see Paterson et al. (4) for a detailed development. For general background on the block-stacking problem, see also Wikipedia (5).

Endpoint reduction (clearance idea). If a block is supported along an overlap interval $[L, R]$, then any vertical support distribution on $[L, R]$ with total force W and total moment Wx^* is equivalent (for equilibrium) to two endpoint forces at L and R:

$$F_L = W\,\frac{R - x^*}{R - L}, \qquad F_R = W\,\frac{x^* - L}{R - L},$$

so that $F_L + F_R = W$ and $LF_L + RF_R = Wx^*$. In the figures below, the *endpoints actually used* are marked explicitly.

2 Example A: three blocks and an overhang of 1

2.1 Geometry

Consider three blocks B_1, B_2, B_3 arranged as follows (see Figure 1):

$$B_1 = [-\tfrac{1}{2},\ \tfrac{1}{2}], \qquad B_2 = [-1,\ 0], \qquad B_3 = [0,\ 1].$$

The rightmost edge of the stack is at $x = 1$, so the overhang (to the right of the table edge $x = 0$) equals 1.

2.2 Center of gravity check

The centers of gravity are

$$\mathrm{cg}(B_1) = 0, \qquad \mathrm{cg}(B_2) = -\tfrac{1}{2}, \qquad \mathrm{cg}(B_3) = \tfrac{1}{2}.$$

Hence the center of gravity of the three-block stack is

$$\mathrm{cg}(B_1 \cup B_2 \cup B_3) = \frac{0 + (-\frac{1}{2}) + (\frac{1}{2})}{3} = 0.$$

The bottom block touches the table only on $[-\frac{1}{2}, 0]$. Since the overall center of gravity lies exactly at $x = 0$, the configuration is *balanced* (critically balanced).

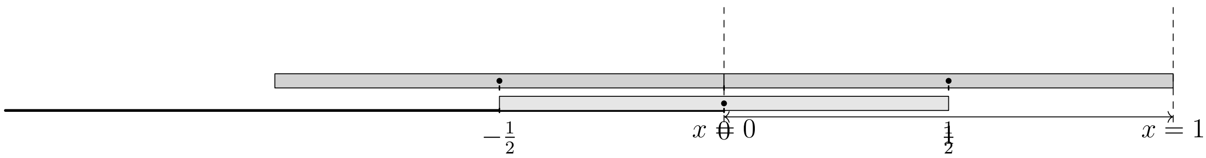


Figure 1: Three blocks with overhang 1. Dots indicate centers of gravity. Tick marks label the overlap endpoints used for endpoint reduction.

3 Example B: four blocks and the Ainley overhang $\frac{15-4\sqrt{2}}{8}$

3.1 What the picture shows

Figure 1 in Paterson et al. (4) reproduces a four-block configuration credited to Ainley (1). It achieves the overhang

$$D = \frac{15 - 4\sqrt{2}}{8} \approx 1.16789,$$

slightly larger than 7/6 (the best overhang obtainable by the symmetric “bridge” arrangement).

3.2 Exact coordinates of the blocks

Let the table edge be at $x = 0$. Define four blocks (bottom, left-middle, right-middle, top) by their left endpoints:

$$B_1 = [x_1,\ x_1 + 1], \quad x_1 = -\frac{1 + 2\sqrt{2}}{8},$$

$$B_2 = [x_2,\ x_2 + 1], \quad x_2 = -\frac{51}{56} - \frac{5\sqrt{2}}{28},$$

$$B_3 = [x_3,\ x_3 + 1], \quad x_3 = \frac{7}{8} - \frac{\sqrt{2}}{2},$$

$$B_4 = [x_4,\ x_4 + 1], \quad x_4 = -\frac{103}{56} + \frac{13\sqrt{2}}{14}.$$

Numerically:

$$x_1 \approx -0.47855, \quad x_2 \approx -1.16325, \quad x_3 \approx 0.16789, \quad x_4 \approx -0.52609.$$

Overlaps (contact regions).

- B_2 rests on B_1 along $[x_1,\ x_2 + 1]$.
- B_3 rests on B_1 along $[x_3,\ x_1 + 1]$.
- B_4 rests on B_2 along $[x_4,\ x_2 + 1]$ (touching at $x_2 + 1$).
- B_4 rests on B_3 along $[x_3,\ x_4 + 1]$ (touching at $x = x_3$).

Overhang. The rightmost edge is the right edge of B_3:

$$D = x_3 + 1 = \frac{15 - 4\sqrt{2}}{8} \approx 1.16789.$$

3.3 Force propagation and moment equations

We show equilibrium using only vertical forces; all weights are 1. In a critically balanced solution we may (by endpoint reduction) take all reaction forces to act at endpoints of the overlap intervals. Define the following upward forces (all nonnegative):

- On B_4: an upward force $2 - \sqrt{2}$ at $x = x_2 + 1$ and an upward force $\sqrt{2} - 1$ at $x = x_3$.
- On B_2: an upward force $3 - \sqrt{2}$ at $x = x_1$ (and 0 at $x = x_2 + 1$).
- On B_3: an upward force $\sqrt{2}$ at $x = x_1 + 1$ (and 0 at $x = x_3$).
- On B_1: an upward force 4 at the table edge $x = 0$ (and 0 at $x = x_1$).

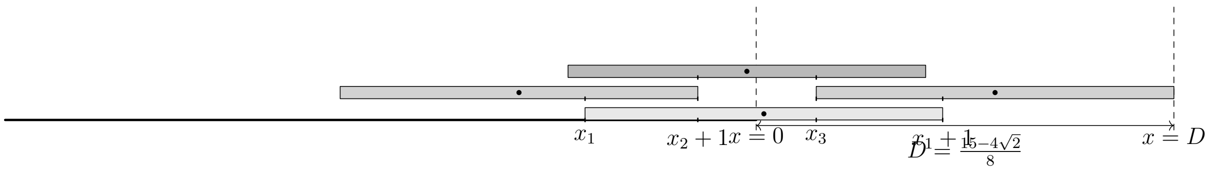


Figure 2: Ainley's four-block configuration (as reproduced in Paterson et al. (4)), drawn from the exact coordinates above. Dots indicate centers of gravity. Tick marks label the overlap endpoints used for endpoint reduction.

4 Remarks on "stability" versus "balance"

The examples above are *balanced*: they satisfy force and moment equilibrium with non-negative vertical reactions. They are typically *critical* (some reactions concentrate at endpoints or even at a single point), so a physical stack needs either friction or a tiny perturbation to become robustly stable. This distinction, and the practice of perturbing balanced stacks to stable ones, is emphasized in Paterson et al. (4).

5 Force collections and the tree viewpoint (Paterson et al., Figure 6)

This section explains the bookkeeping device used in Paterson et al. (4) (Figure 6): one represents the internal balancing forces as a *flow of vertical forces* through the stack. Each block acts as a local "rearrangement gadget": it receives some vertical support forces from below, it *uses up* one unit of force to support its own weight, and it passes the remaining force upward to the blocks above, while preserving the correct moment.

5.1 Forces as a discrete distribution

If block B_i rests on a block (or the table) below, then the support can be represented by a finite set of upward point forces

$$(x_{i,1}, f_{i,1}),\ (x_{i,2}, f_{i,2}),\ \ldots$$

applied along the *bottom edge* of B_i (each $x_{i,t}$ lies in the relevant contact interval). Similarly, B_i applies upward point forces to blocks above it along its *top edge*. Because we ignore friction and shear, all forces are vertical; because we are in static equilibrium, only total force and total moment matter.

A convenient abstraction is to view a set of upward point forces as a discrete "mass distribution"

$$\mu = \{(x_1, m_1), \ldots, (x_k, m_k)\},$$

where the m_j are magnitudes of forces and the x_j are application points.

5.2 The collections F_i and the local conservation laws

Number blocks bottom-to-top. Let B_0 denote the table. For $0 \leq i < n$, let F_i denote the collection of *upward* forces exerted by objects in

$$\{B_0, B_1, \ldots, B_i\}$$

(on the left side of a cut) on blocks in

$$\{B_{i+1}, \ldots, B_n\}$$

(on the right side of a cut). Thus F_0 consists of the forces the table applies to the stack; F_{n-1} consists of the forces supporting only the top block.

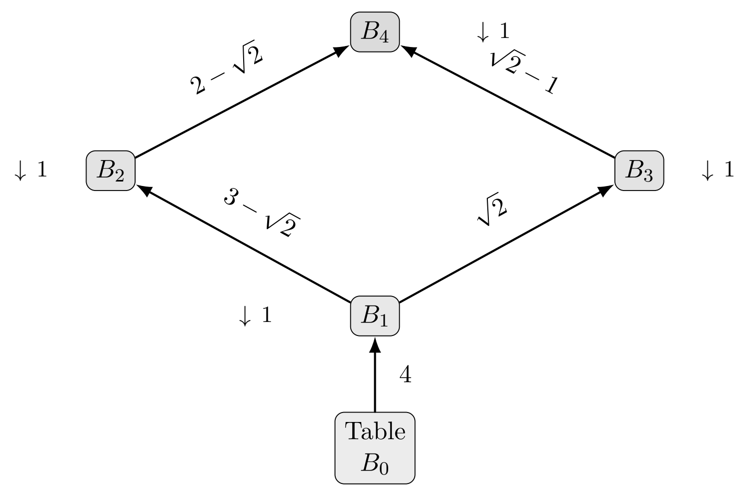
Compare two consecutive cuts F_i and F_{i+1}. Passing the cut from below B_i to above B_i removes the forces that act *upward on* B_i and replaces them with the forces that B_i exerts *upward on blocks above it.* Static equilibrium of B_i yields two equations (total force and total moment). Let a be the left edge of B_i (so B_i spans $[a, a+1]$) and let $\{(x_t, f_t)\}$ be the upward forces applied to B_i from below. Let $\{(x'_s, f'_s)\}$ be the upward forces applied by B_i to blocks above. Then equilibrium of B_i is exactly:

$$\sum_t f_t = 1 + \sum_s f'_s, \tag{1}$$

$$\sum_t x_t f_t = \left(a + \tfrac{1}{2}\right) + \sum_s x'_s f'_s. \tag{2}$$

5.3 A 4-book tree example (Ainley configuration)

Figure 3 gives a compact force-tree for the four-book Ainley stack. Edge labels are transmitted vertical force magnitudes. Moment balance determines *where along the contact intervals* these forces act (as shown by the endpoints in Figure 2).



Edge labels = transmitted vertical forces. At each block: incoming force = outgoing force +1 (its weight).

Figure 3: A force-tree for the four-book Ainley stack: magnitudes only; locations are fixed by moment balance (endpoints in Figure 2).

6 A nonlinear programming view and a KKT check for Ainley's 4-block stack

This section reformulates the four-block configuration as a small nonlinear program (NLP). The nonlinearity comes from *bilinear* moment terms "position × force". By endpoint reduction, it suffices to represent each contact by forces acting at *endpoints* of overlap intervals.

6.1 Decision variables

Blocks are $B_i = [x_i, x_i + 1]$ with $x_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$. We introduce endpoint reaction forces (all *upward* and nonnegative):

- Table $\to B_1$ on $[x_1, 0]$: forces t_L at $x = x_1$ and t_R at $x = 0$.
- $B_1 \to B_2$ on $[x_1, x_2 + 1]$: forces f_{12}^L at $x = x_1$ and f_{12}^R at $x = x_2 + 1$.
- $B_1 \to B_3$ on $[x_3, x_1 + 1]$: forces f_{13}^L at $x = x_3$ and f_{13}^R at $x = x_1 + 1$.
- $B_2 \to B_4$ on $[x_4, x_2 + 1]$: forces f_{24}^L at $x = x_4$ and f_{24}^R at $x = x_2 + 1$.
- $B_3 \to B_4$ on $[x_3, x_4 + 1]$: forces f_{34}^L at $x = x_3$ and f_{34}^R at $x = x_4 + 1$.

All forces satisfy

$$t_L, t_R, f_{12}^L, f_{12}^R, f_{13}^L, f_{13}^R, f_{24}^L, f_{24}^R, f_{34}^L, f_{34}^R \;\geq\; 0.$$

6.2 Objective (maximize overhang)

Introduce an overhang variable D and maximize it subject to being at least the right edge of each block:

$$\max\; D \qquad \text{s.t.} \qquad D \;\geq\; x_i + 1 \quad (i = 1, 2, 3, 4).$$

In Ainley's layout, the optimal rightmost edge is attained by B_3, so at the optimum one has $D = x_3 + 1$.

6.3 Equilibrium constraints (force and moment for each block)

All weights are 1, acting at $x_i + \frac{1}{2}$.

Top block B_4.

$$f_{24}^L + f_{24}^R + f_{34}^L + f_{34}^R = 1, \tag{3}$$

$$x_4 + \tfrac{1}{2} = x_4 f_{24}^L + (x_2 + 1) f_{24}^R + x_3 f_{34}^L + (x_4 + 1) f_{34}^R. \tag{4}$$

Block B_2 (supported by B_1, supports B_4).

$$f_{12}^L + f_{12}^R = 1 + (f_{24}^L + f_{24}^R), \tag{5}$$

$$(x_2 + \tfrac{1}{2}) + x_4 f_{24}^L + (x_2 + 1) f_{24}^R = x_1 f_{12}^L + (x_2 + 1) f_{12}^R. \tag{6}$$

Block B_3 (supported by B_1, supports B_4).

$$f_{13}^L + f_{13}^R = 1 + (f_{34}^L + f_{34}^R), \tag{7}$$

$$(x_3 + \tfrac{1}{2}) + x_3 f_{34}^L + (x_4 + 1) f_{34}^R = x_3 f_{13}^L + (x_1 + 1) f_{13}^R. \tag{8}$$

Bottom block B_1 (supported by table, supports B_2 and B_3).

$$t_L + t_R = 1 + (f_{12}^L + f_{12}^R) + (f_{13}^L + f_{13}^R), \tag{9}$$

$$(x_1 + \tfrac{1}{2}) + x_1 f_{12}^L + (x_2 + 1) f_{12}^R + x_3 f_{13}^L + (x_1 + 1) f_{13}^R = x_1 t_L + 0 \cdot t_R. \tag{10}$$

6.4 Geometry constraints (intended contact pattern)

To encode the Ainley-type contact pattern, impose linear overlap constraints such as:

$$x_1 \le 0 \le x_1 + 1$$

(bottom block crosses the table edge), together with the overlap relations listed earlier in Example B (e.g., $x_2 \le x_1 \le x_2 + 1 \le x_1 + 1$, $x_1 \le x_3 \le x_1 + 1 \le x_3 + 1$, etc.). At Ainley's optimum these inequalities are satisfied and (for KKT purposes) are not active in the simplest check below.

6.5 KKT conditions and verification for Ainley's solution

Write the problem in minimization form by minimizing $-D$. Let $\lambda_1, \dots, \lambda_8$ be multipliers for the eight equilibrium equalities (3)–(10). Let $\mu \ge 0$ be multipliers for the nonnegativity constraints written as $-u \le 0$.

The KKT conditions are:

- **Primal feasibility:** all equalities, inequalities and nonnegativity constraints hold.
- **Dual feasibility:** $\mu \ge 0$ for all inequality multipliers.
- **Complementary slackness:** $\mu_u \, u = 0$ for each force variable u.
- **Stationarity:** $\nabla(-D) + \sum_{k=1}^{8} \lambda_k \nabla E_k + \sum_u \mu_u \nabla(-u) = 0$, where E_k denotes the kth equilibrium equality.

Ainley's point. The coordinates are

$$x_1 = -\frac{1 + 2\sqrt{2}}{8}, \quad x_2 = -\frac{51}{56} - \frac{5\sqrt{2}}{28}, \quad x_3 = \frac{7}{8} - \frac{\sqrt{2}}{2}, \quad x_4 = -\frac{103}{56} + \frac{13\sqrt{2}}{14},$$

with $D = x_3 + 1 = \frac{15 - 4\sqrt{2}}{8}$. The endpoint forces used in Example B are

$$t_L = 0, \quad t_R = 4,$$

$$f_{12}^L = 3 - \sqrt{2}, \quad f_{12}^R = 0, \qquad f_{13}^L = 0, \quad f_{13}^R = \sqrt{2},$$
$$f_{24}^L = 0, \quad f_{24}^R = 2 - \sqrt{2}, \qquad f_{34}^L = \sqrt{2} - 1, \quad f_{34}^R = 0.$$

Direct substitution shows that all equilibrium equalities (3)–(10) hold exactly, and all forces are nonnegative.

Active set and complementary slackness. The active nonnegativity constraints are precisely

$$t_L = f_{12}^R = f_{13}^L = f_{24}^L = f_{34}^R = 0.$$

All other forces are strictly positive, hence their inequality multipliers can be set to 0.

A valid set of KKT multipliers. One choice of equality multipliers (in the order of (3),(4),(5),(6),(7),(8),(9),(10)) is

$$\lambda_1 = \frac{1}{32}+\frac{\sqrt{2}}{16},\ \lambda_2 = 0,\ \lambda_3 = \frac{1}{32}+\frac{\sqrt{2}}{16},\ \lambda_4 = 0,\ \lambda_5 = -\frac{15}{32}+\frac{\sqrt{2}}{2},\ \lambda_6 = \frac{\sqrt{2}}{2},\ \lambda_7 = 0,\ \lambda_8 = \frac{1}{4}.$$

For the active inequality multipliers (in the order $t_L, f_{12}^R, f_{13}^L, f_{24}^L, f_{34}^R$), one may take

$$\mu_{t_L} = \frac{1}{32}+\frac{\sqrt{2}}{16},\qquad \mu_{12R} = \frac{3}{56}+\frac{\sqrt{2}}{56},\qquad \mu_{13L} = \frac{1}{4}-\frac{\sqrt{2}}{16},\qquad \mu_{24L} = 0,\qquad \mu_{34R} = \frac{10}{7}-\frac{6\sqrt{2}}{7},$$

which are all nonnegative. With these multipliers, the stationarity equation holds for the objective $\min(-D)$ at Ainley's point, and complementary slackness holds because each listed μ corresponds to a force that is 0.

Conclusion. Ainley's four-block configuration is a KKT point for the endpoint-force NLP model above (a necessary condition for local optimality in this nonconvex formulation).

7 Companion implementation and coordinate mapping

The companion Pygame implementation (2) uses the same endpoint-reduction viewpoint: for each book it computes the resultant load location x^*, then routes the total load to two bracketing overlap endpoints. The code uses pixel coordinates; the mapping to this report is:

$$X \ = \ X_{\text{edge}} + (\text{block-length}) \cdot x,$$

where X_{edge} is the pixel position of the table edge and the block-length is the pixel width of one book. In a built-in test mode, the code reproduces Example A and Example B directly from the (x_i) given above.

8 References

References

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